Advanced Digital Signal Processing

Part 6: Multi-Rate Digital Signal Processing

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Contents

Basic Ideas

Why multi-rate systems?

- In many practical signal processing applications ***different sampling rates*** are present, corresponding to different bandwidths of the individual signals $\Longrightarrow$ ***multi-rate systems***.
- Often a signal has to be converted from one rate to another. This process is called ***sampling rate conversion***.
    - Sampling rate conversion can be carried out by ***analog means***, that is D/A conversion followed by A/D conversion using a different sampling rate
      $\Longrightarrow$ D/A converter introduces signal distortion, and the A/D converter leads to quantization effects.
    - Sampling rate conversion can also be carried out completely in the ***digital domain***:
      Less signal distortions, more elegant and efficient approach.
- $\Longrightarrow$ Topic of this chapter is multi-rate signal processing and sampling rate conversion in the digital domain.

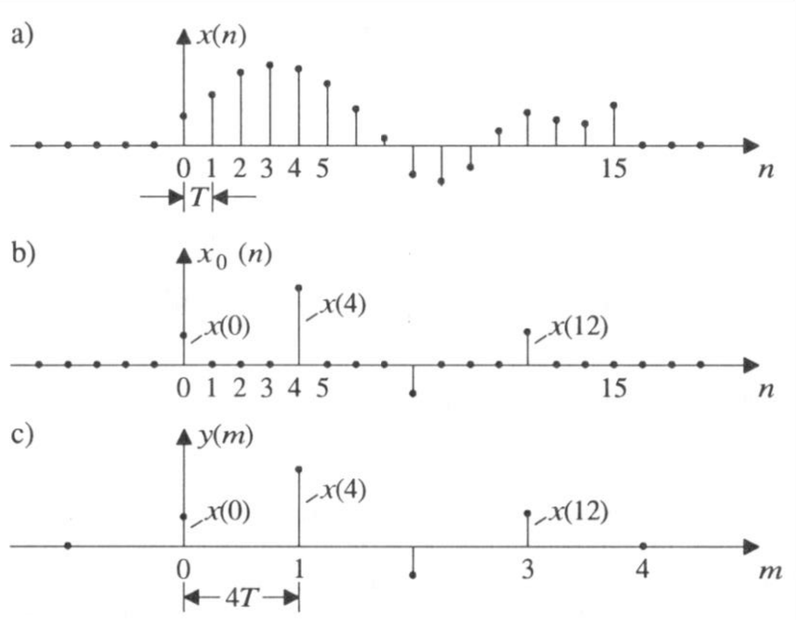
Basic Multi-Rate Operations – Part 1

Sampling rate reduction – Part 1:

Reduction of the sampling rate (downsampling) by a factor M: Only every M-th value of the signal $x(n)$ is used for further processing, i.e. $y(m) = x(m \cdot M)$.

$$x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow y(m)$$

Example: Sampling rate reduction by factor 4



Some kind of intermediate signal that is used for easier understanding of the equations that will follow!

From [Fliege: Multiraten-Signalverarbeitung, 1993]

Sampling rate reduction – Part 1:

Spectrum after downsampling – Part 1:

In the z-domain we have

$$X_0(z) = \sum_{n=-\infty}^{\infty} x_0(n)\, z^{-n}$$

… Inserting the definition of the signal $x_0(n)$ ***and exploiting that*** $x_0(n)$ ***contains a lot of zeros …***

$$= \sum_{m=-\infty}^{\infty} x(mM)\, z^{-mM}$$

… inserting the definition of $y(n)$ ***…***

$$= \sum_{m=-\infty}^{\infty} y(m)\, \left[z^M\right]^{-m}$$

… inserting the definition of the z-transform …

$$= Y(z^M).$$

Sampling rate reduction – Part 2:

Spectrum after downsampling – Part 2:

Starting point: orthogonality of the complex exponential sequence

$$\frac{1}{M}\sum_{k=0}^{M-1} e^{j2\pi km/M} = \begin{cases} 1, & \text{for } m = \lambda M, \quad \lambda \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

With $x_0(mM) = x(mM)$ it follows

$$x_0(m) = x(m)\,\frac{1}{M}\sum_{k=0}^{M-1} W_M^{-km} \quad \text{with} \quad W_M = e^{-j2\pi/M}.$$

Inserting the result from above

The z-transform $X_0(z)$ can be obtained as

$$X_0(z) = \sum_{m=-\infty}^{\infty} x_0(m)\, z^{-m}$$

… rearranging the sums and inserting the definition of the z-transform …

$$= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{m=-\infty}^{\infty} x(m)\left[z\,W_M^k\right]^{-m} = \frac{1}{M}\sum_{k=0}^{M-1} X\big(z\,W_M^k\big).$$

Sampling rate reduction – Part 3:

Spectrum after downsampling – Part 3:

By replacing $Y(z^M) = X_0(z)$ in the last equation we have for the z-transform of the downsampled sequence $y(m)$

$$\boxed{Y(z^M) \;=\; \frac{1}{M}\sum_{k=0}^{M-1} X(zW_M^k).}$$

With $z = e^{j\Omega}$ and $\Omega' = \Omega M$ the corresponding spectrum can be derived from

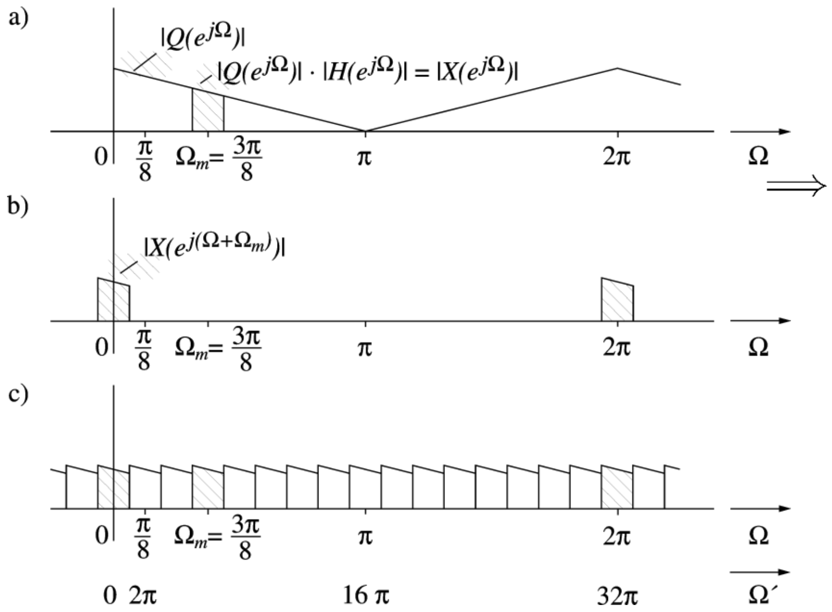
$$\boxed{Y(e^{j\Omega'}) \;=\; \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\Omega'-k2\pi)/M)}\right).}$$

$\Longrightarrow$ ***Downsampling by factor** M **leads to a periodic repetition of the spectrum** $X(e^{j\Omega})$ **at intervals of** $2\pi/M$*
(related to the high sampling frequency).

Sampling rate reduction – Part 5:

Frequency response after downsampling – Part 3:

Example: Sampling rate reduction of a bandpass signal by $M = 16$



(a) Bandpass spectrum $X(e^{j\Omega})$ is obtained by filtering.

(b) Shift to the baseband, followed by decimation with $M = 16$

(c) Magnitude frequency response $|X(e^{j\Omega})|$ at the lower sampling rate.

From [Vary, Heute, Hess: Digitale Sprachsignalverarbeitung, 1998]

Remark: Shifted versions of $X(e^{j\Omega})$ are weighted with the factor $1/M$ according to the last slide.

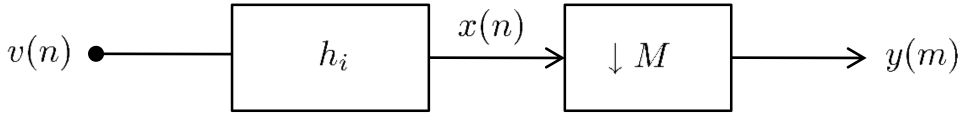
Basic Multi-Rate Operations – Part 6

Sampling rate reduction – Part 6:

Decimation and aliasing – Part 1:

If the sampling theorem is violated in the lower clock rate, we obtain *spectral overlapping* between the repeated spectra
$\Longrightarrow$ This is called ***aliasing***.

How to avoid aliasing? ***Band limitation*** of the input signal prior to the sampling rate reduction with an anti-aliasing filter h_i (lowpass filter).



$\Longrightarrow$ Anti-aliasing filtering followed by downsampling is often called ***decimation***.

Specification for the desired magnitude frequency response of the lowpass anti-aliasing (or decimation) filter:

$$\left|H_{\mathrm{des}}(e^{j\Omega})\right| = \begin{cases} 1, & \text{if} \quad |\Omega| + \lambda 2\pi \leq \frac{\Omega_{\mathrm{c}}}{M}, \\ 0, & \text{else}, \end{cases}$$

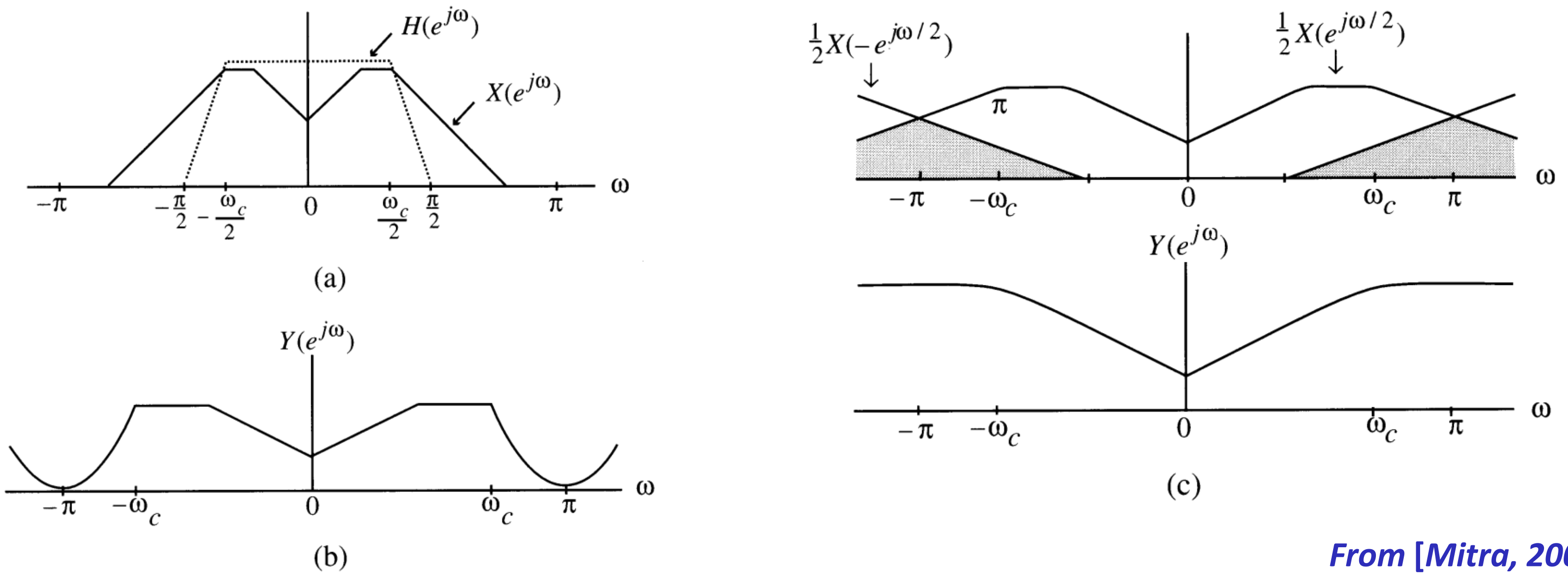
where $\Omega_{\mathrm{c}} < \pi$ denotes the highest frequency that needs to be preserved in the ***decimated*** signal.

Sampling rate reduction – Part 7:

Decimation and aliasing – Part 2:

Downsampling in the frequency domain, illustration for $M = 2$:

- (a) input filter spectra, $(V \rightarrow X,\ \Omega \rightarrow \omega)$
- (b) output of the decimator,
- (c) no filtering, only downsampling



From [Mitra, 2000]

Questions

Questions about sample rate reduction:

Partner work – Please think about the following questions and try to find answers (first group discussions, afterwards broad discussion in the whole group).

- What happens in the spectral domain when you decimate (without filtering) the time-domain signal?

  ……

  ……

- Is an anti-aliasing filter always necessary? If not, what are the conditions for applying such a filter?

  ……

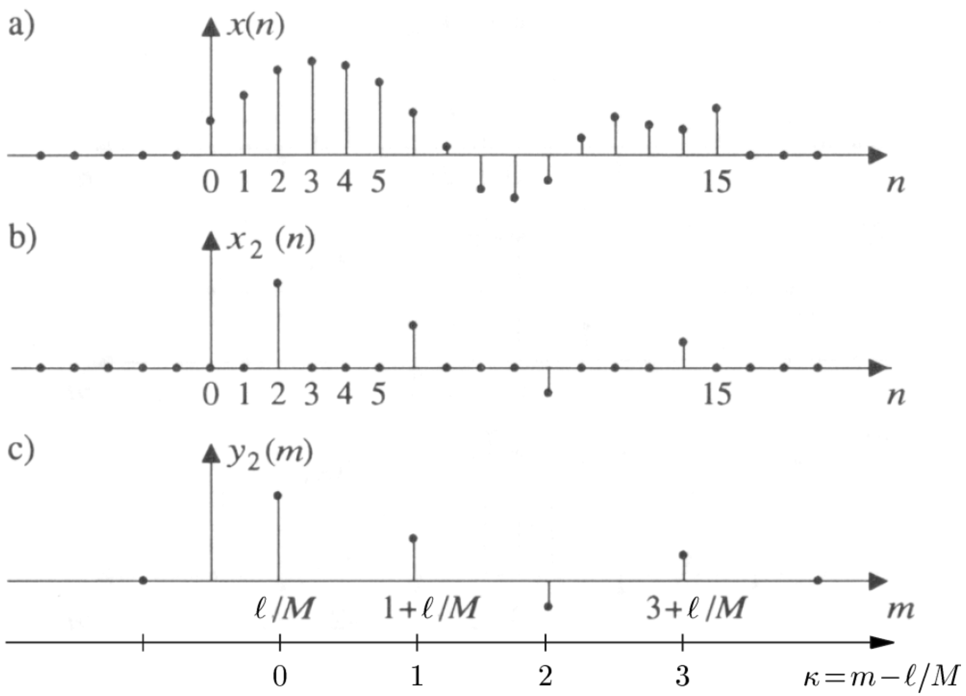
  ……

Sampling rate reduction – Part 8:

More general approach: sampling rate reduction with phase offset – Part 1:

Up to now we have always used $y(0) = x(0)$, now we introduce an additional phase offset l into the decimation process.

Example for $l = 2$



From [Fliege: Multiraten-Signal-verarbeitung, 1993]

Sampling rate reduction – Part 9:

More general approach: sampling rate reduction with phase offset – Part 2:

Derivation of the Fourier transform of the output signal $y(m)$:

Orthogonality relation of the complex exponential sequence:

$$\frac{1}{M}\sum_{k=0}^{M-1} e^{j\frac{2\pi}{M}k(m-l)} = \begin{cases} 1, & \text{if } m = \lambda M + l, \quad \lambda \in \mathbb{Z}, \\ 0, & \text{else.} \end{cases}$$

Using that we have

$$x_l(m) = x(m)\,\frac{1}{M}\sum_{k=0}^{M-1} W_M^{-k(m-l)},$$

and transforming that into the z-domain yields

$$\begin{aligned} X_l(z) &= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{m=-\infty}^{\infty} x(m)\left(W_M^k\, z\right)^{-m} W_M^{kl} \\ &= \frac{1}{M}\sum_{k=0}^{M-1} X(zW_M^k)\, W_M^{kl}. \end{aligned}$$

Sampling rate reduction – Part 10:

More general approach: sampling rate reduction with phase offset – Part 3:

The frequency response can be obtained from the last equation by substituting $z = e^{j\Omega}$ and $\Omega' = M\Omega$ as

$$
\begin{aligned}
Y_l(e^{j\Omega'}) &= \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\frac{\Omega'-2\pi k}{M}}\right) W_M^{kl}, \\
Y_l(e^{jM\Omega}) &= \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\Omega-\frac{2\pi k}{M})}\right) W_M^{kl}.
\end{aligned}
$$

$\Longrightarrow$ We can see that each repeated spectrum is weighted with a complex exponential (rotation) factor (called “twiddle” factor).

Basic Multi-Rate Operations – Part 11

Sampling rate increase – Part 1:

Increase of the sampling rate by factor L *(**upsampling**)*:

$$y(m) \longrightarrow \boxed{\uparrow L} \longrightarrow u(n)$$

Insertion of $L-1$ zeros samples between all samples of $y(m)$

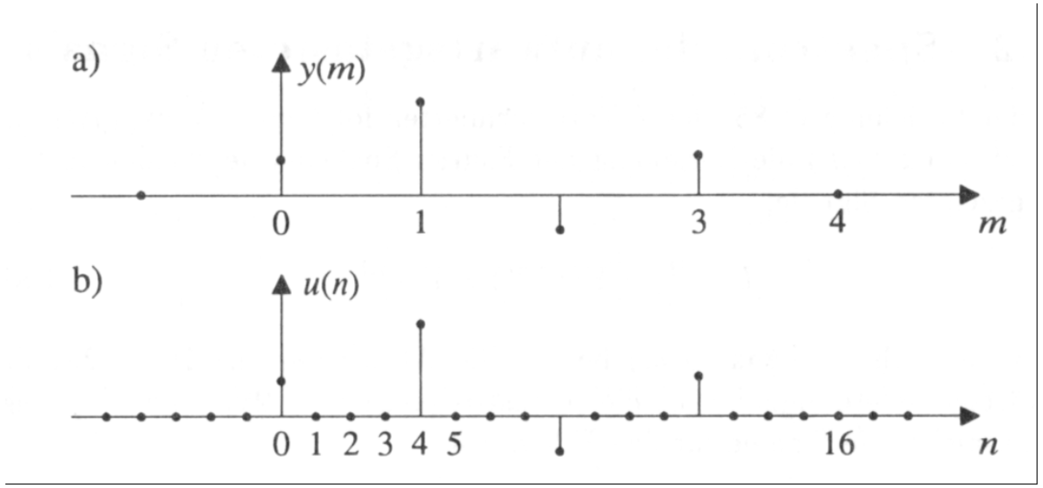
$$u(n) = \begin{cases} y\left(\frac{n}{L}\right), & \text{if } n = \lambda L, \quad \lambda \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}$$

Notation: Since the upsampling factor is named with L in conformance with the majority of the technical literature in the following we will denote the *length for an FIR filter* with L_F.

Basic Multi-Rate Operations – Part 12

Sampling rate increase – Part 2:

Example: Sampling rate increase by factor 4



From [Fliege: Multiraten-Signalverarbeitung, 1993]

In the z-domain the input/output relation is

$$\boxed{U(z) \; = \; Y(z^L).}$$

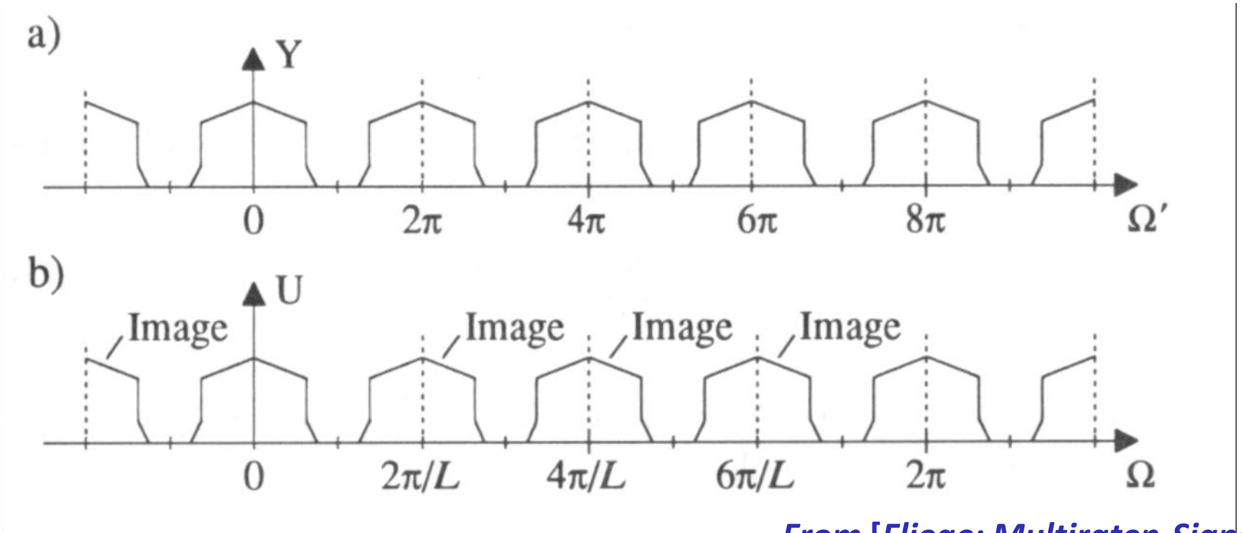
Sampling rate increase – Part 3:

Frequency response after upsampling:

From the last equation we obtain with $z = e^{j\Omega}$

$$\boxed{U(e^{j\Omega}) \;=\; Y(e^{jL\Omega}).}$$

$\Longrightarrow$ The frequency response of $y(m)$ does not change by upsampling, however the frequency axis is scaled differently. The new sampling frequency is now (in terns of Ω' for the lower sampling rate) equal to $L \cdot 2\pi$.



From [Fliege: Multiraten-Signalverarbeitung, 1993]

Basic Multi-Rate Operations – Part 14

Sampling rate increase – Part 4:

Interpolation – Part 1:

The inserted zero values are interpolated with suitable values which corresponds to the suppression of the $L-1$ ***imaging spectra*** in the frequency domain by a suitable lowpass interpolation filter.

$$y(m) \longrightarrow \boxed{\uparrow L} \xrightarrow{u(n)} \boxed{g_i} \longrightarrow v(n)$$

g_i : Interpolation or ***anti-imaging*** lowpass filter

Basic Multi-Rate Operations – Part 15

Sampling rate increase – Part 5:

Interpolation – Part 2:

Specifications for the interpolation filter:

Suppose $y(m)$ is obtained by sampling a bandlimited continuous-time signal $y_{\mathrm{a}}(t)$ at the Nyquist rate (such that the sampling theorem is just satisfied). The Fourier transform can thus be written with $\omega = \Omega/T_0$ as

$$Y(e^{j\Omega}) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} Y_{\mathrm{a}}\left(\frac{j(\Omega - 2\pi k)}{T_0}\right),$$

where T_0 denotes the sampling period. If we instead sample $y_{\mathrm{a}}(t)$ at a much higher rate $T = T_0/L$ we have

$$\begin{aligned} V(e^{j\Omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} Y_{\mathrm{a}}\left(\frac{j(\Omega - 2\pi k)}{T}\right), \\ &= \frac{L}{T_0} \sum_{k=-\infty}^{\infty} Y_{\mathrm{a}}\left(\frac{j(\Omega - 2\pi k)}{(T_0/L)}\right). \end{aligned}$$

Sampling rate increase – Part 6:

Interpolation – Part 3:

On the other hand by upsampling of $y(m)$ with factor L we obtain the Fourier transform of the upsampled sequence $u(n)$ analog to the first equation of the last slide as

$$U(e^{j\Omega}) \;=\; Y(e^{j\Omega L}).$$

$\Longrightarrow$ If $u(n)$ is passed through an ideal lowpass filter with cut-off frequency π/L and a gain of L, the output of the filter will be precisely $v(n) = \mathcal{F}^{-1}\{V(e^{j\Omega})\}$.

Therefore, we can now state our specifications for the lowpass interpolation filter:

$$\left|H_{\text{des}}(e^{j\Omega})\right| \;=\; \begin{cases} L, & \text{if} \quad |\Omega| + \lambda 2\pi \;\leq\; \frac{\Omega_c}{L}, \\ 0, & \text{else.} \end{cases}$$

Where Ω_c denotes the highest frequency that needs to be preserved in the interpolated signal (related to the lower sampling frequency).

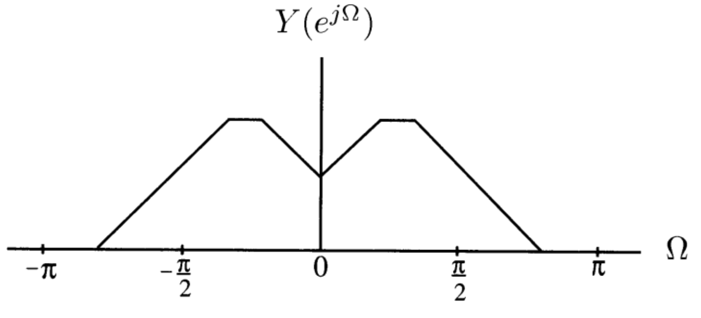
Basic Multi-Rate Operations – Part 17

Sampling rate increase – Part 7:

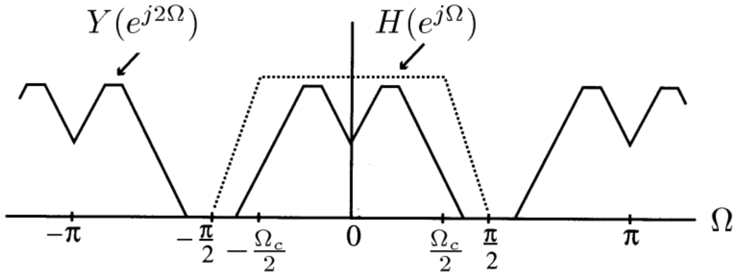
Interpolation – Part 4:

Upsampling in the frequency domain, illustration for $L = 2$:

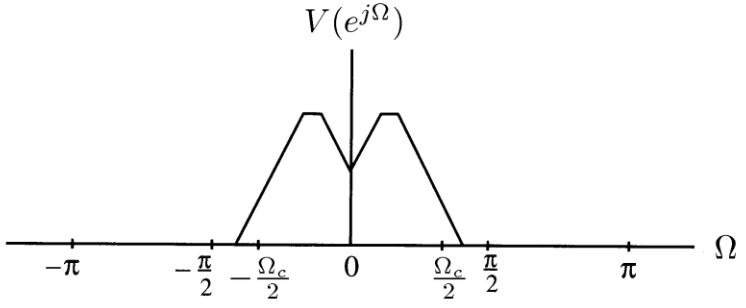
(a) Input spectrum, (b) output of the upsampler, (c) output after interpolation with the filter h_i



(a)



(b)



(c)

From [Mitra, 2000]

Basic Multi-Rate Operations – Part 18

Example: Decimation and interpolation – Part 1:

Consider the following structure:

$$x(n) \longrightarrow \boxed{h_i} \xrightarrow{u(n)} \boxed{\downarrow M} \xrightarrow{y(m)} \boxed{\uparrow M} \xrightarrow{v(n)} \boxed{f_i} \longrightarrow \hat{x}(n)$$

Input-output relation?

Basic Multi-Rate Operations – Part 19

Example: Decimation and interpolation – Part 2:

Relation between $Y(z)$ and $U(z)$, where z is replaced by $z^{1/M}$:

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} U(z^{1/M} W_M^k),$$

which by using $U(z) = H(z)\, X(z)$ leads to

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(z^{1/M} W_M^k)\, X(z^{1/M} W_M^k).$$

With $V(z) = Y(z^M)$ it follows

$$V(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(z W_M^k)\, X(z W_M^k),$$

And we finally have

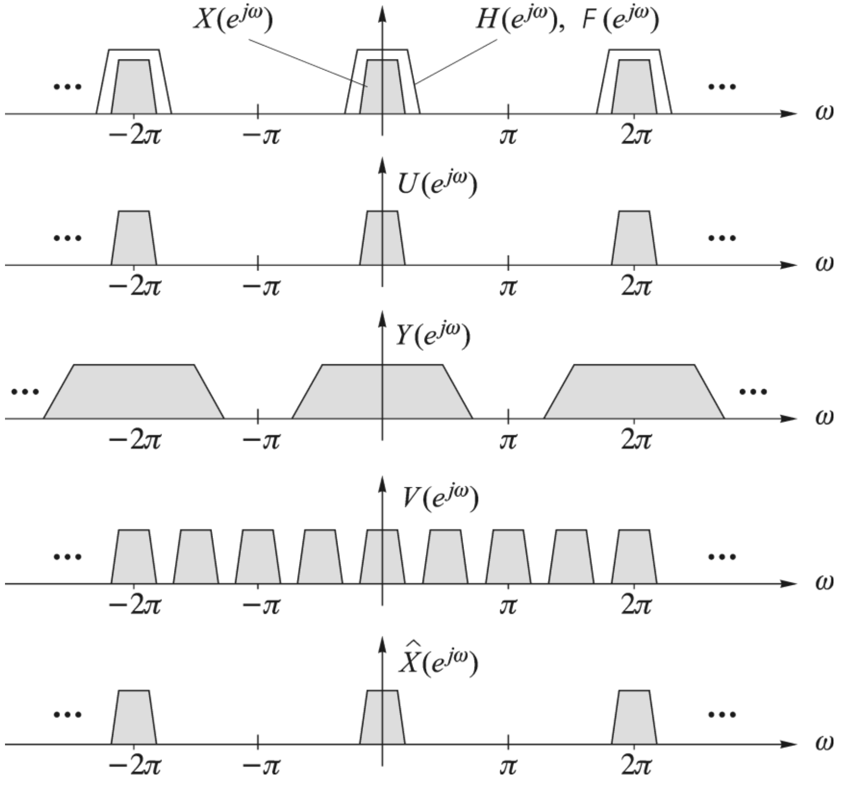
$$\boxed{\hat{X}(z) = F(z)\, Y(z^M) = \frac{1}{M} \sum_{k=0}^{M-1} F(z)\, H(z W_M^k)\, X(z W_M^k).}$$

Basic Multi-Rate Operations – Part 20

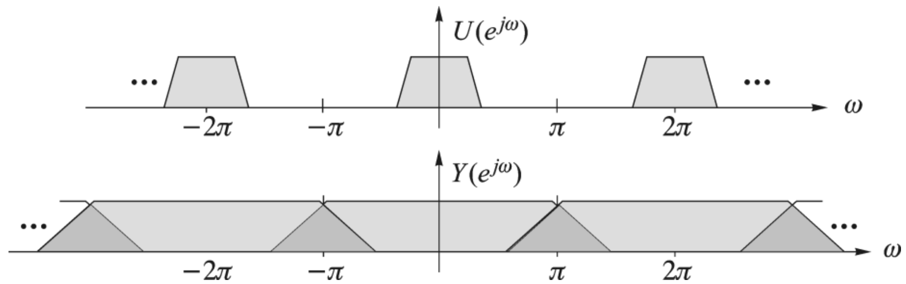
Example: Decimation and interpolation – Part 3:

Example $(\omega \rightarrow \Omega)$

$M = 4$, no aliasing:



with aliasing:



From [Mertins: Signal Analysis, 1999]

Questions

Motivation of multi-rate structures

Partner work – Please think about the following questions and try to find answers
(first group discussions, afterwards broad discussion in the whole group).

- If you would like to convolve a signal at a sample rate of 10 kHz with an impulse response (FIR filter) of 10 seconds length, how many multiplications and additions do you need per second?

  ………

  ………

- Assume that you can split the signal into 10 equally wide bandpass signals (assmuming that you have ideal filters that are “for free”) and you can use the largest possible subsampling rate, how many multiplications and additions do you need now (again per second)?

  ………

  ………

Basic Multi-Rate Operations – Part 21

Polyphase decomposition – Part 1:

A polyphase decomposition of a sequence $x(n)$ leads to M subsequences $x_l(m),\ l = 0, ..., M-1$ which contain only every M-th value of $x(n)$.

Example for $M = 2$:

Decomposition into an ***even*** and ***odd*** subsequence.

This is an important tool for the derivation of efficient multi-rate filtering structures (as we will see later on).

Three different decomposition types:

- ***Type-1 polyphase components***:

  Decomposition of $x(n)$ into $x_l(m),\ l = 0, ..., M-1$ with

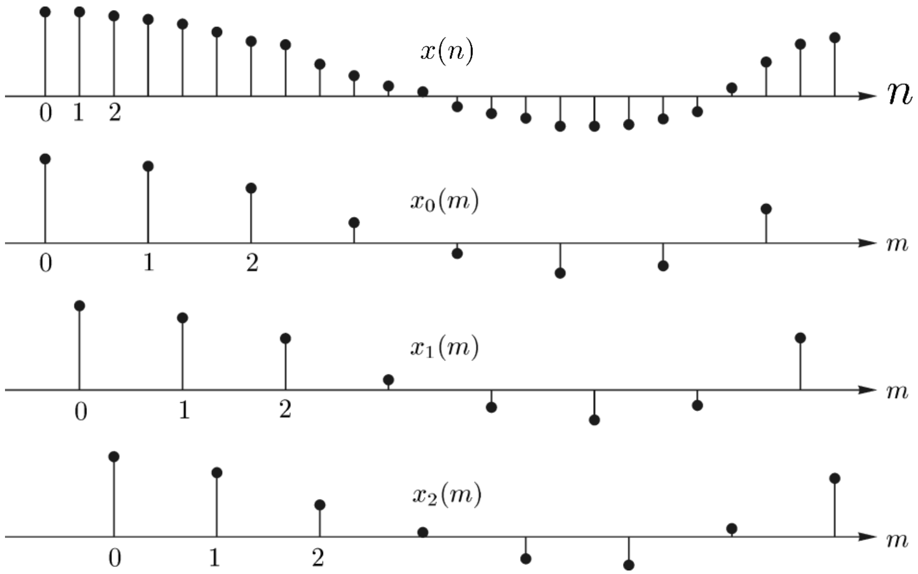
  $$x_l(m) \;=\; x(mM+l), \quad n = mM+l,\ m \in \mathbb{Z}.$$

  With $x_l(m) \circ\!\!-\!\!\bullet\ X_l(z)$ the z-transform $X(z)$ can be obtained as

  $$\boxed{X(z) \;=\; \sum_{l=0}^{M-1} z^{-l}\, X_l(z^M).}$$

Polyphase decomposition – Part 2:

Example for $M = 3$



$$
\begin{aligned}
x_0(0) &= x(0), \quad & x_0(1) &= x(3), \quad & \ldots \\
x_1(0) &= x(1), \quad & x_1(1) &= x(4), \quad & \ldots \\
x_2(0) &= x(2), \quad & x_2(1) &= x(5), \quad & \ldots
\end{aligned}
$$

Basic Multi-Rate Operations – Part 23

Polyphase decomposition – Part 3:

- *Type-2 polyphase components*:

$$\boxed{X(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} X_l'(z^M)}$$

with $X_l'(z) \bullet\!\!-\!\!\circ\; x_l'(n) = x(nM + M - 1 - l)$

Example for $M = 3$

$$\begin{aligned} x_0'(0) &= x(2), \quad & x_0'(1) &= x(5), \quad & \ldots \\ x_1'(0) &= x(1), \quad & x_1'(1) &= x(4), \quad & \ldots \\ x_2'(0) &= x(0), \quad & x_2'(1) &= x(3), \quad & \ldots \end{aligned}$$

Polyphase decomposition – Part 4:

- *Type-3 polyphase components*:

$$\boxed{X(z) \;=\; \sum_{l=0}^{M-1} z^{l}\,\bar{X}_l(z^M)}$$

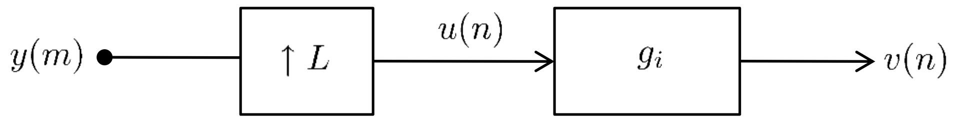
with $\bar{X}_l(z) \;\bullet\!\!-\!\!\!-\!\!\circ\; \bar{x}_l(n) \;=\; x(nM-l)$

Basic Multi-Rate Operations – Part 25

Nyquist-Filters – Part 1:

Nyquist- or L-band filters:

- Used as interpolator filters since they ***preserve the nonzero samples*** at the output of the upsampler ***also at the interpolator output***.
- ***Computationally more efficient*** since they contain zero coefficients.
- Preferred in interpolator and decimator designs.



The input-output relation of the interpolator can be stated as $V(z) = G(z)Y(z^L)$.

The filter $G(z)$ can be written in polyphase notation according to

$$G(z) \;=\; G_0(z^L) + z^{-1}\, G_1(z^L) + \cdots + z^{-(L-1)}\, G_{L-1}(z^L),$$

Where $G_l(z)$, $l = 0,\; ...,\; L-1$ denote the type 1 polyphase components of the filter $G(z)$.

Basic Multi-Rate Operations – Part 26

Nyquist-Filters – Part 2:

Suppose now that the k^{th} polyphase component of $G(z)$ is a constant, i.e. $G_k(z) = \alpha$.
Then the interpolator output $V(z)$ can be expressed as

$$V(z) = \alpha\, z^{-k}\, Y(z^L) + \sum_{l=0,\, l\neq k}^{L-1} z^{-l}\, G_l(z^L)\, Y(z^L).$$

$\Rightarrow v(Ln+k) = \alpha y(n);$ the input samples appear at the output of the system without any distortion for all n.
All in-between $(L-1)$ samples are determined by interpolation.

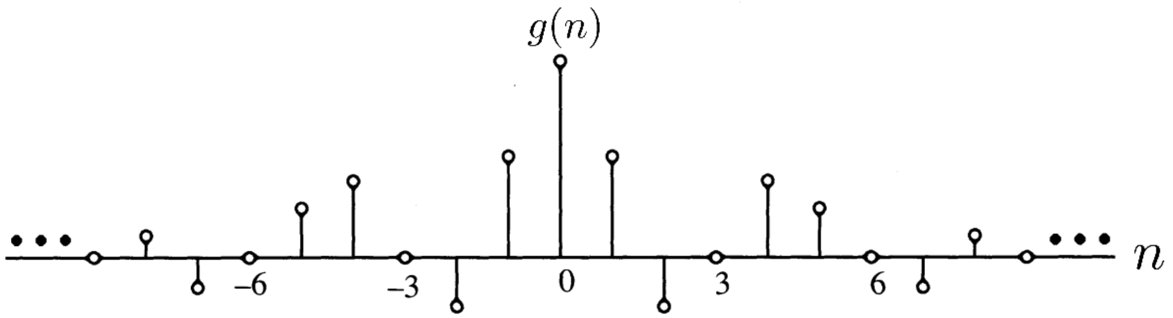
Nyquist-Filters – Part 3:

Properties

- Impulse response of a zero-phase L-th band filter:

$$g_{Ln} = \begin{cases} \alpha & \text{for} \quad n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\Longrightarrow$ every L-th coefficient is zero (except for $n = 0$) $\Longrightarrow$ computationally attractive



From [Mitra, 2000]

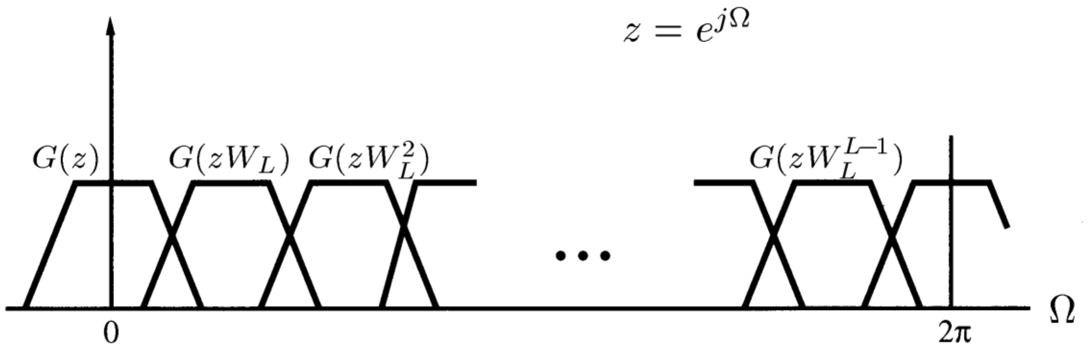
Nyquist-Filters – Part 4:

Properties

- It can be shown for $\alpha = 1/L$ that for a zero-phase L-th band filter:

$$\sum_{l=0}^{L-1} G(zW_L^l) \;=\; L\alpha \;=\; 1.$$

The sum of all L uniformly shifted version of $G(e^{j\Omega})$ add up to a constant.



From [Mitra, 2000]

Questions

Questions about sample filterbanks:

Partner work – Please think about the following question and try to find answers (first group discussions, afterwards broad discussion in the whole group).

- Please try to derive the equation

$$\sum_{l=0}^{L-1} G(zW_L^l) \;=\; 1, \quad \text{with } g_0 \;=\; 1/L \text{ and } g_{iL} \;=\; 0$$

by transforming the equation first to the Fourier domain and afterwards to the time domain.

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Basic Multi-Rate Operations – Part 29

Nyquist-Filters – Part 5:

Half-band filters:

Special case of L-band filters for $L = 2$

- Transfer function $G(z) = \alpha + z^{-1}G_1(z^2)$
- For $\alpha = 1/2$ we have for the zero-phase filter g_i

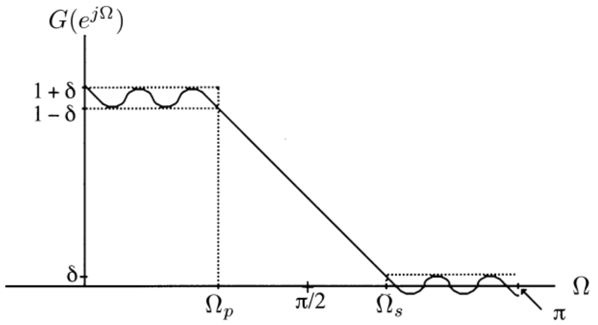
$$G(z) + G(-z) = 1.$$

- If g_i is real-valued then $G(-e^{j\Omega}) = G(e^{j(\pi-\Omega)})$ and it follows

$$\boxed{G(e^{j\Omega}) + G(e^{j(\pi-\Omega)}) \quad = \quad 1.}$$

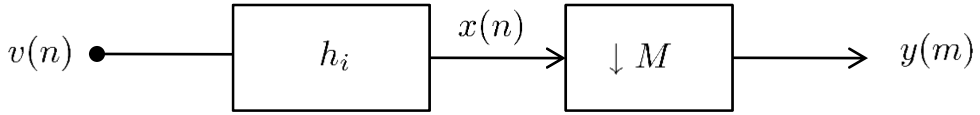
$\Rightarrow G(e^{j\Omega})$ exhibits a symmetry with respect to the half-band frequency $\pi/2 \rightarrow$ ***halfband filter.***

- FIR linear phase halfband filter: Length is restricted to $L_F = 4\lambda - 1,\ \lambda \in \mathbb{N}$



From [Mitra, 2000]

FIR direct form realization for decimation – Part 1:



The convolution with the length L_F FIR Filter h_i can be described as

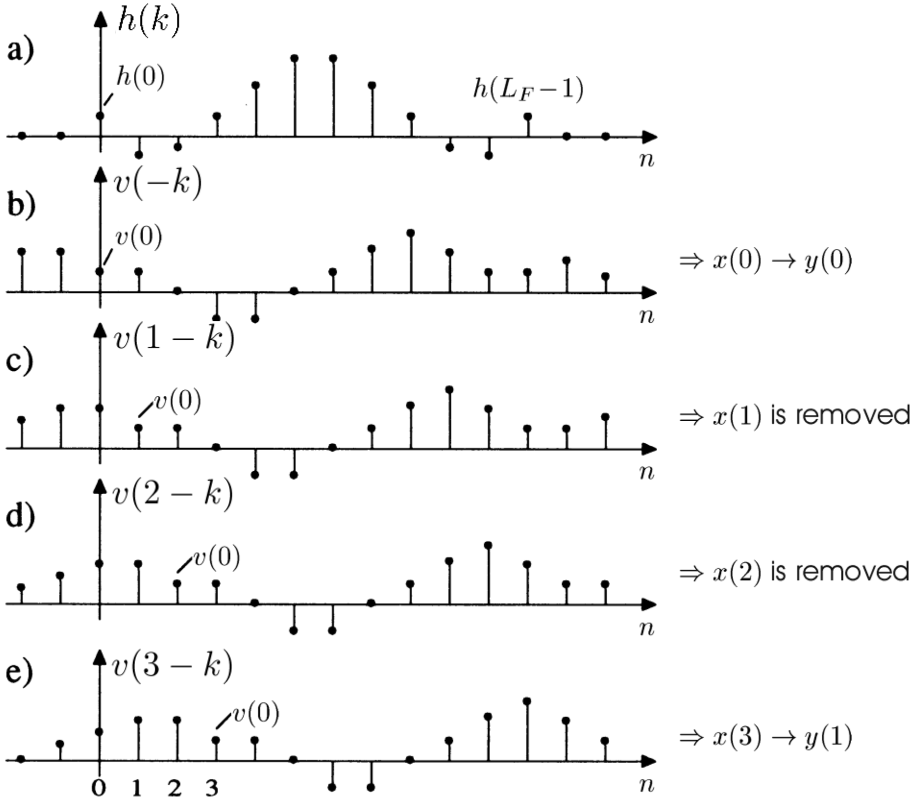
$$x(n) = \sum_{i=0}^{L_F-1} h_i \, v(n-i)$$

and the downsampling as $y(m) = x(m\,M)$. Combining both equations we can write the decimation operation according to

$$y(m) = \sum_{i=0}^{L_F-1} h_i \, v(mM-i).$$

FIR direct form realization for decimation – Part 2:

Visualization $(M = 3)$:



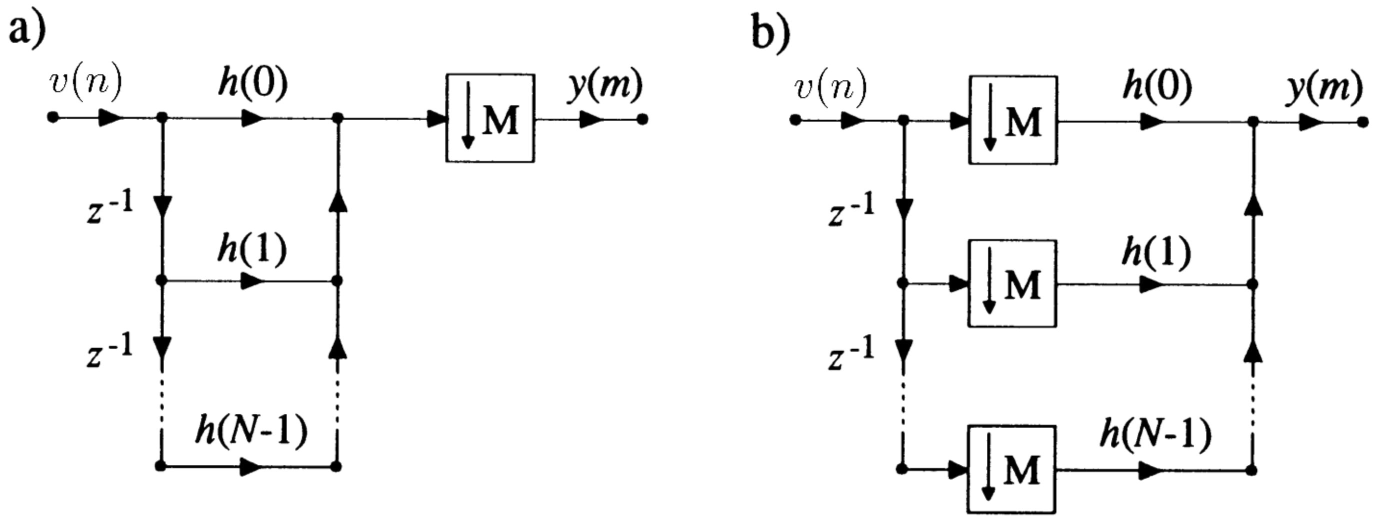
→ Multiplication of h_i with $v(1-i)$ and $v(2-i)$ leads to the result $x(1)$ and $x(2)$ which are discarded in the decimation process

→ these compositions are not necessary.

Structures for Decimation and Interpolation – Part 3

FIR direct form realization for decimation – Part 3:

More efficient implementation $(L_F \rightarrow N)$:



From [Fliege: Multiraten-Signalverarbeitung, 1993]

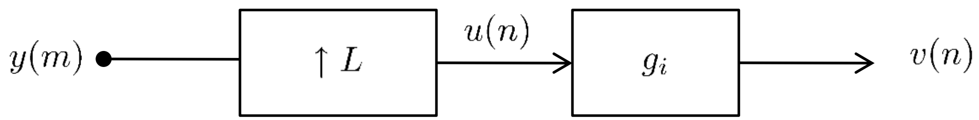
(a) Antialiasing FIR filter in first direct form followed by downsampling.

(b) Efficient structure obtained from ***shifting the downsampler before the multipliers***:

- Multiplications and additions are now performed at the lower sampling rate.
- Additional reductions can be obtained by exploiting the symmetry of h_i (linear-phase).

Structures for Decimation and Interpolation – Part 4

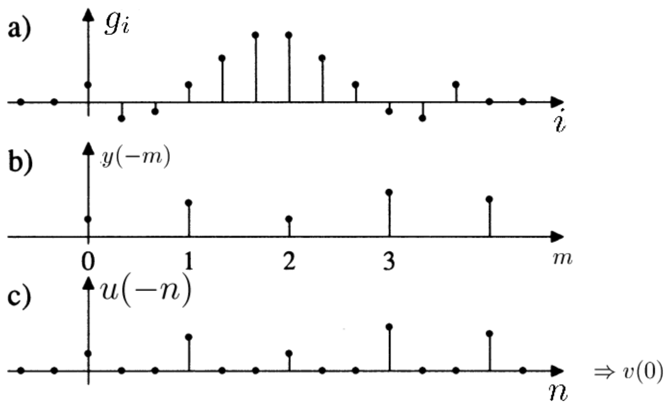
FIR direct form realization for interpolation – Part 1:



The output $v(n)$ of the interpolation filter can be obtained as convolution with the length

$$v(n) \;=\; \sum_{i=0}^{L_F-1} g_i \cdot u(n-i),$$

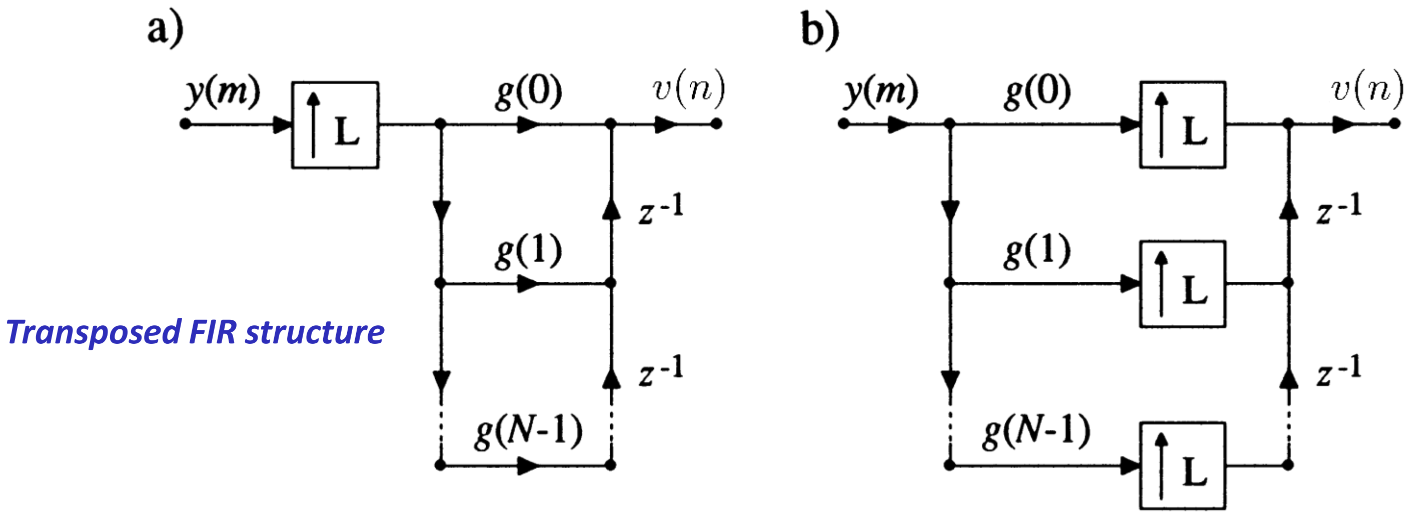
Which is depicted in the following:



→ The output sample $v(0)$ is obtained by multiplication of g_i with $u(-i)$, where a lot of zero multiplications are involved, which are inserted by upsampling operation.

FIR direct form realization for interpolation – Part 2:

More efficient implementation $(L_F \rightarrow N)$:



(a) Upsampling followed by interpolation FIR filter in second direct form

(b) Efficient structure obtained from ***shifting the upsampler behind the multipliers***:

- Multiplications are now performed at the lower sampling rate, however the output delay chain still runs in the higher sampling rate.
- Zero multiplications are avoided.
- Additional reductions can be obtained by exploiting the symmetry of h_i (linear-phase).

Decimation and Interpolation with Polyphase Filters – Part 1

Decimation – Part 1:

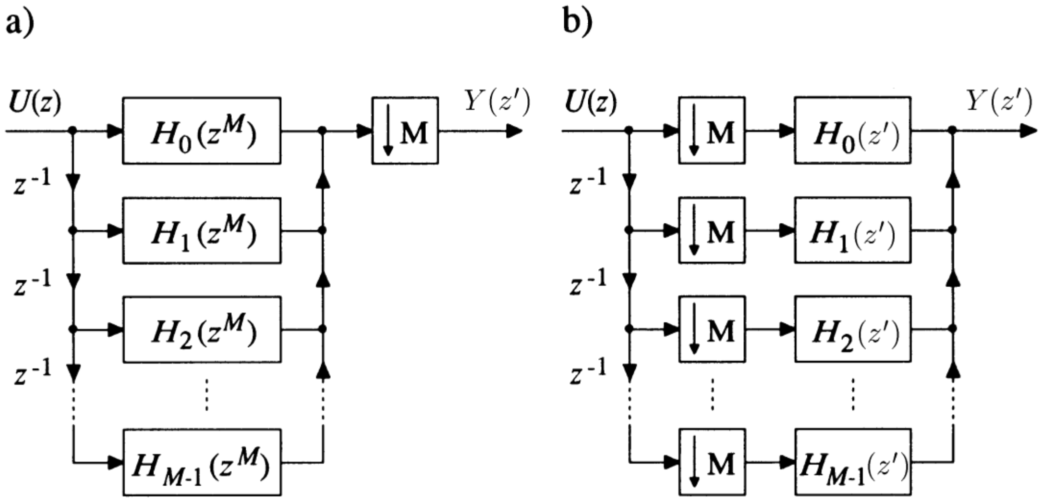
- From previous sections we know that a sequence can be decomposed into polyphase components. Here type-1 polyphase components are considered in the following.
- Type-1 polyphase decomposition of the decimation filter h_i : The z-transform $H(z)$ can be written as

$$H(z) \;=\; \sum_{l=0}^{M-1} z^{-l}\, H_l(z^M),$$

M denoting the downsampling factor and $H_l(z') \;\bullet\!\!-\!\!\circ\; h_l(m)$ the z-transform for type-1 polyphase components $h_l(m)$, $l \;=\; 0,\, ...,\, M-1.$

Decimation – Part 2:

Resulting decimator structure $(V(z) \rightarrow U(z))$:



From [Fliege: Multiraten-Signalverarbeitung, 1993]

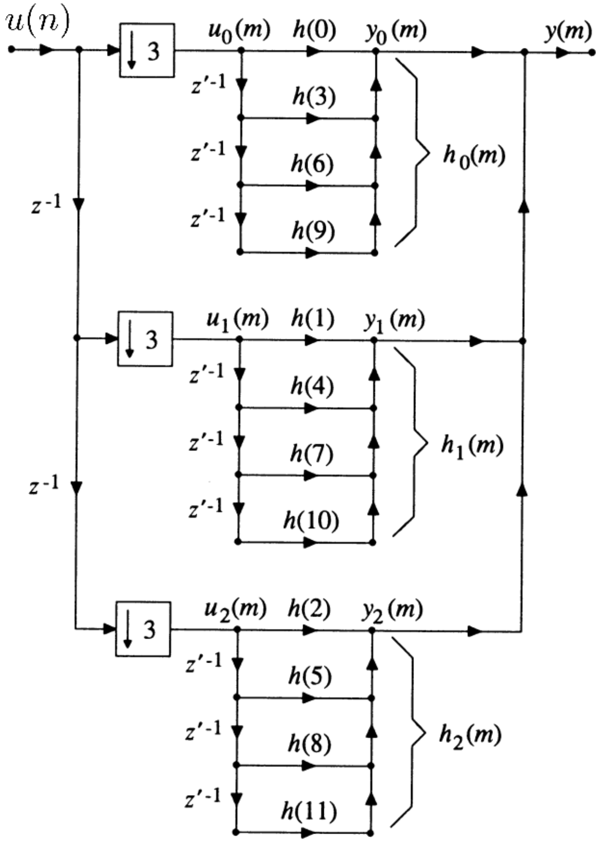
(a) Decimator with decimation filter in polyphase representation

(b) Efficient version of (a) with M times reduced complexity

Remark: The structure (b) has the same complexity as the direct form structure from the previous section, therefore no further advantage. However, the polyphase structures are important for digital filter banks.

Decimation – Part 3:

Structure (b) in time domain $(v(n) \rightarrow u(n))$:



From [Fliege: Multiraten-Signalverarbeitung, 1993]

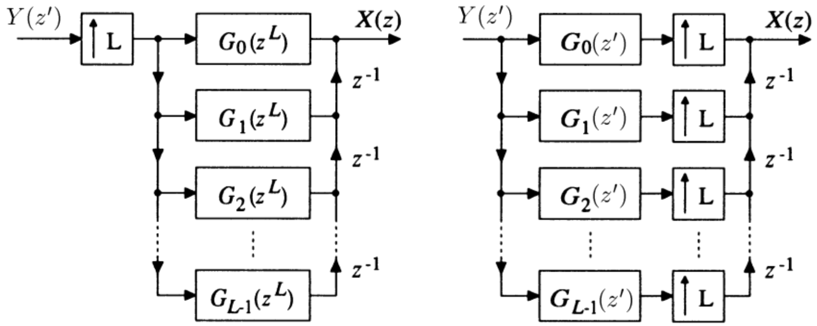
Interpolation – Part 1:

Transfer function of the interpolation filter can be written for the decimation filter as

$$G(z) = \sum_{l=0}^{L-1} z^l \, G_l(z^L),$$

L denoting the upsampling factor, and $g_l(m)$ the type-1 polyphase components of g_i with $g_l(m) \circ\!\!-\!\!\bullet\, G_l(z')$.

Resulting interpolator structure $(V(z) \rightarrow X(z))$:



From [Fliege: Multiraten-Signalverarbeitung, 1993]

(a) Interpolator with interpolation filter in polyphase representation

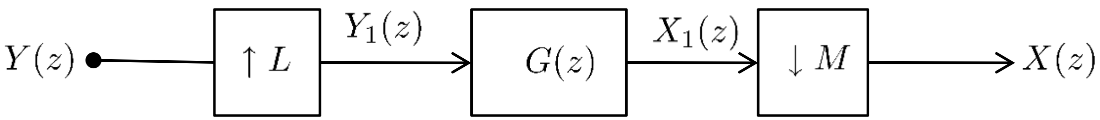
(b) Efficient version of (a) with L times reduced complexity

As in the decimator case the computational complexity of the efficient structure is the same as for the direct form interpolation from the previous section.

Non-Integer Sampling Rate Conversion – Part 1

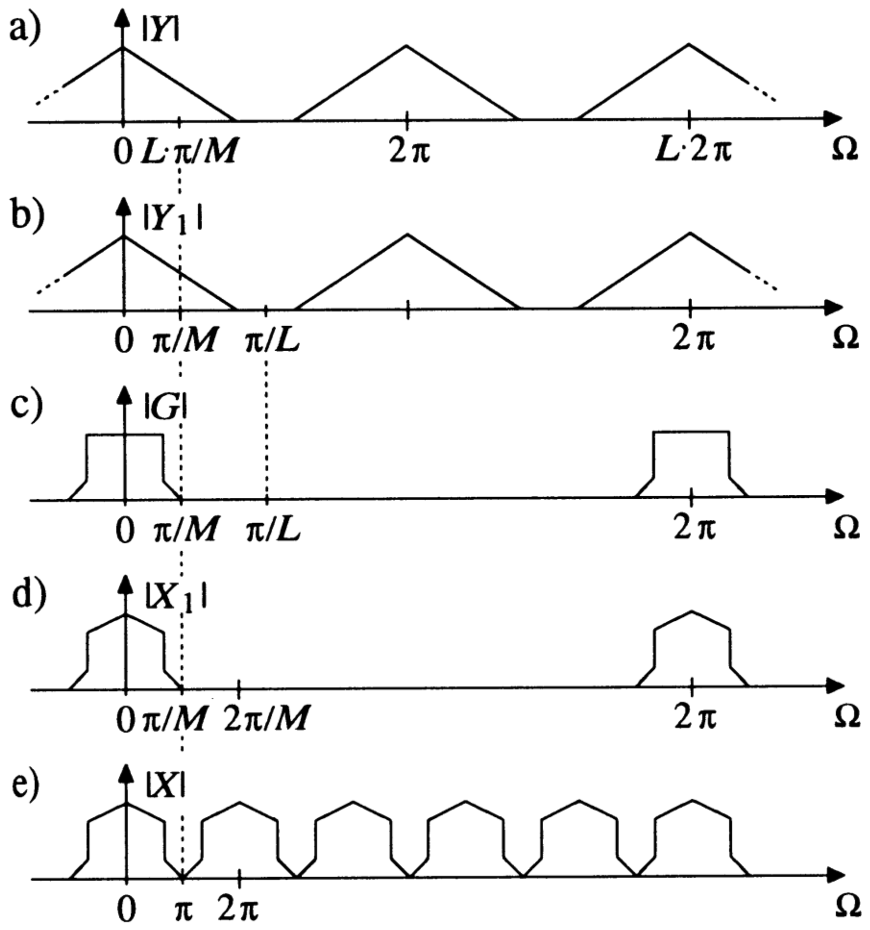
Notation: For simplicity a delay by one sample will be generally denoted with z^{-1} for every sampling rate in a multi-rate system in the following (instead of introducing a special z for each sampling rate as in the sections before).

- In practice often there are applications where data has to be ***converted between different sampling rates with a rational ratio***.
- ***Non-integer (synchronous) sampling rate conversion*** by factor L/M :
  Interpolation by factor L, followed by a decimation by factor M; decimation and interpolation filter can be combined:



Non-Integer Sampling Rate Conversion – Part 2

- Magnitude frequency responses:

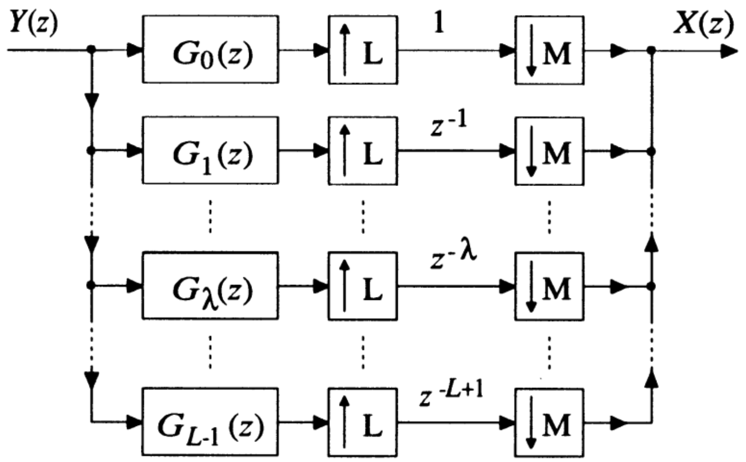


From [Fliege: Multiraten-Signalverarbeitung, 1993]

Efficient conversion structure – Part 1:

In the following derivation of the conversion structure we assume a ratio $L/M < 1$. However, a ration $L/M > 1$ can also be used with *dual* structures.

1. Implementation of the filter $G(z)$ in polyphase structure, shifting of all subsamplers into the polyphase branches:



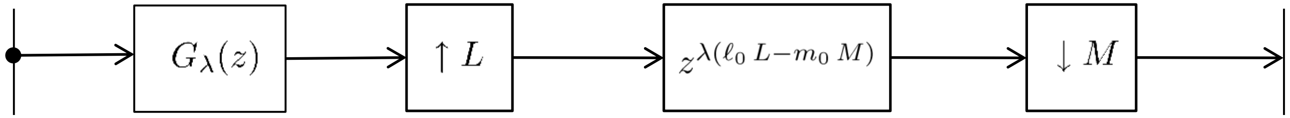
From [Fliege: Multiraten-Signalverarbeitung, 1993]

Efficient conversion structure – Part 1:

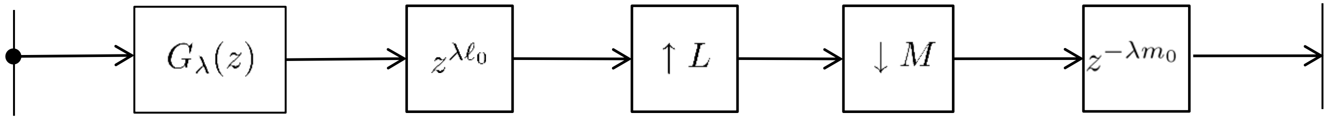
2. Application of the following structural simplifications:
   a. It is known that if L and M are coprime (that is they have no common divider except one) we can find such that $\ell_0, m_0 \in \mathbb{N}$

$$\ell_0\, L - m_0\, M = -1 \quad \text{(diophantic equation)}$$

→ delay $z^{-\lambda}$ in one branch of the polyphase structure can be replaced with the delay $z^{\lambda(\ell_0\, L - m_0\, M)}$



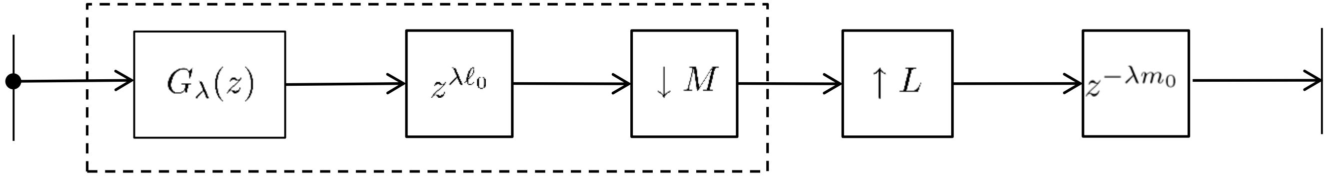
   b. The factor $z^{\lambda \ell_0 L}$ can be shifted before the upsampler, and the factor $z^{-\lambda m_0 M}$ behind the downsampler:



Non-Integer Sampling Rate Conversion – Part 5

Efficient conversion structure – Part 2:

2. Application of the following structural simplifications:
   c. Finally, if M and L are coprime, it can be shown that up- and downsampler may be exchanged in their order:

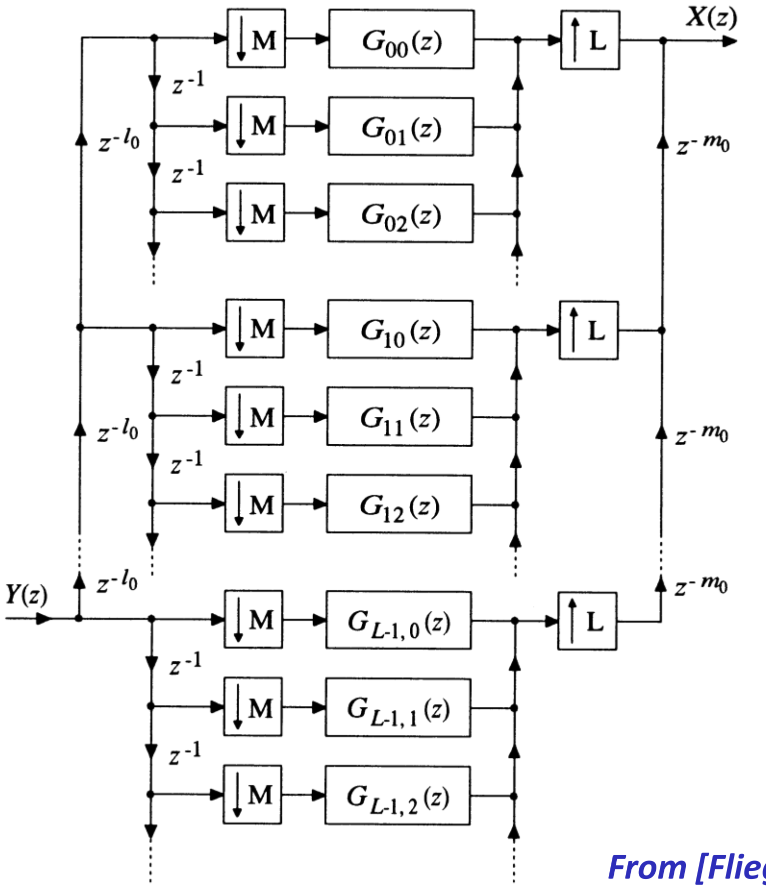


   d. In every branch we now have a decimator (marked with the dashed box), which can again be efficiently realized using the polyphase structure from the previous section. Thus, each type-1 polyphase component $g_\lambda(n)$ is itself decomposed again in M polyphase components $g_{\lambda\mu}(n) \circ\!\!-\!\!\bullet\, G_{\lambda\mu}(z)$, $\lambda = 0, ..., L-1$, $\mu = 0, ..., M-1$.

Efficient conversion structure – Part 3:

Resulting structure:

$$\frac{L}{M} < 1$$



From [Fliege: Multiraten-Signalverarbeitung, 1993]

Non-Integer Sampling Rate Conversion – Part 7

Efficient conversion structure – Part 4:

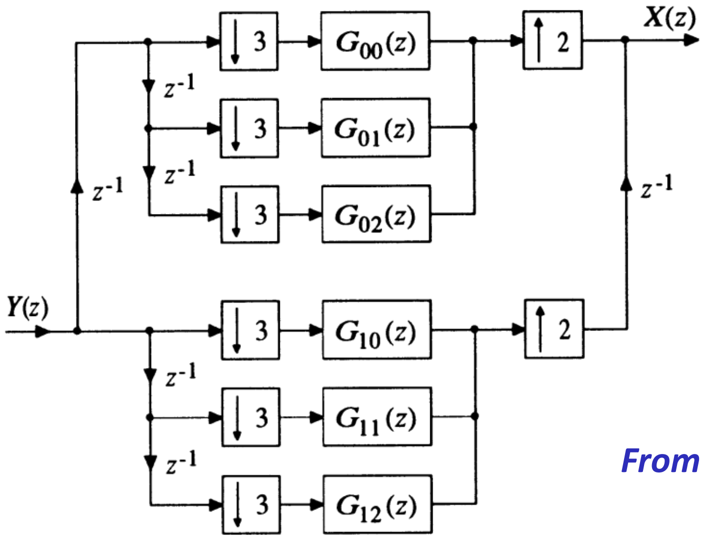
- Delays $z^{-\lambda m_0}$ are realized with the output delay chain.
- The terms $z^{\lambda \ell_0}$ are non-causal elements: In order to obtain a causal representation, we have to insert the extra delay block $z^{-(L-1)\ell_0}$ at the input of the whole system, which cancels out the “negative“ delays $z^{\lambda \ell_0}$.
- Polyphase filters are calculated with the lowest possible sampling rate.
- $L/M > 1$ is realized using the dual structure (exchange: input $\leftrightarrow$ output, downsamplers $\leftrightarrow$ upsamplers, summation points $\leftrightarrow$ branching points, reverse all branching directions)

(transposition with exchanging up and down sampling)

Efficient conversion structure – Part 5:

Example for $L = 2$ and $M = 3$:

Application: Sampling rate conversion for digital audio signals from 48 kHz to 32 kHz sampling rate



From [Fliege: Multiraten-Signalverarbeitung, 1993]

Polyphase filters are calculated with 16 kHz sampling rate compared to 48 kHz sampling rate in the original structure.

Rate conversion from 32 kHz to 48 kHz: Exercise!

Summary – Part 1

- Introduction
- Digital processing of continuous-time signals
- DFT and FFT
- Digital filters
- ***Multi-rate digital signal processing***
  - ***Decimation and interpolation***
  - ***Filters in sampling rate alteration systems***
  - ***Polyphase decomposition and efficient structures***

Summary – Part 2

- Introduction
- Digital processing of continuous-time signals
- Efficient FIR structures
- DFT and FFT
- Digital filters
- Multi-rate digital signal processing

Summary – Part 3

And finally:

Enjoy applying your new knowledge – in the upcoming lectures, during a lab, while working on your thesis and most importantly during your profession as an engineer.

The DSS team