ADSP - Exercise 1: Model Solution and Teaching Notes

Notation as in the lecture notes, Part 2: ω analog angular frequency, $\omega_\mathrm{s} = 2\pi/T$, $\Omega = \omega T$, $V_\mathrm{a}(j\omega)$ input spectrum with highest frequency ω_n, $V_\mathrm{i}(j\omega)$ spectrum of the ideally sampled signal, H_sh sample-and-hold circuit, H_r reconstruction filter, $\hat{y}_\mathrm{a}(t)$ reconstructed output, $\mathrm{si}(x) = \sin(x)/x$.

Teaching focus and timing

Work format: students first think individually (about 90 s for the reactivation questions), then discuss in plenary; pairs are used only in the oral examination. The teacher operates the notebook; no programming is required. Numerical calculation is not the focus; expect sketches, equations, and verbal explanations. The reactivation block may be shortened if the basics are secure.

Suggested 45-minute teaching sequence (the times are not printed on the worksheet):

Phase	Time	Teaching purpose
1. Reactivation: sampling and aliasing	0-12 min	diagnostic block; may be shortened if the basics are secure
2. Demo prediction and observation	12-22 min	sampling rate, anti-aliasing options, aliasing of the 6 kHz tone
3. The complete AD-DSP-DA chain	22-35 min	frequency mapping, digital filter, sample-and-hold, equivalent system
4. Peer oral examination	35-43 min	3 + 3 min examination, 2 min feedback
5. Take-away	43-45 min	secure the two central statements

Planned and extra tasks. The worksheet is ordered by topic and does not mark this distinction, because all tasks are exam-relevant. The 45-minute plan covers the tasks in the middle column; the extra tasks are for self-study, and the answers below cover them.

Section	Planned in class	Extra (self-study, exam-relevant)
1 Reactivation: sampling and aliasing	1.1, 1.2 (may be shortened if the basics are secure)	–
2 Demo prediction and observation	questions 1-3	–
3 The complete AD-DSP-DA chain	3.1, 3.2, 3.3, 3.4	–

1. Reactivation: sampling and aliasing (0-12 min)

The original spectrum is a symmetric triangle supported on $[-\omega_\mathrm{n}, \omega_\mathrm{n}]$. Ideal sampling repeats this spectrum at every integer multiple of ω_s. The factor $1/T$ changes only the common amplitude scale and is irrelevant for the requested overlap sketches.

1.1 $\omega_\mathrm{s} = 3\omega_\mathrm{n}$

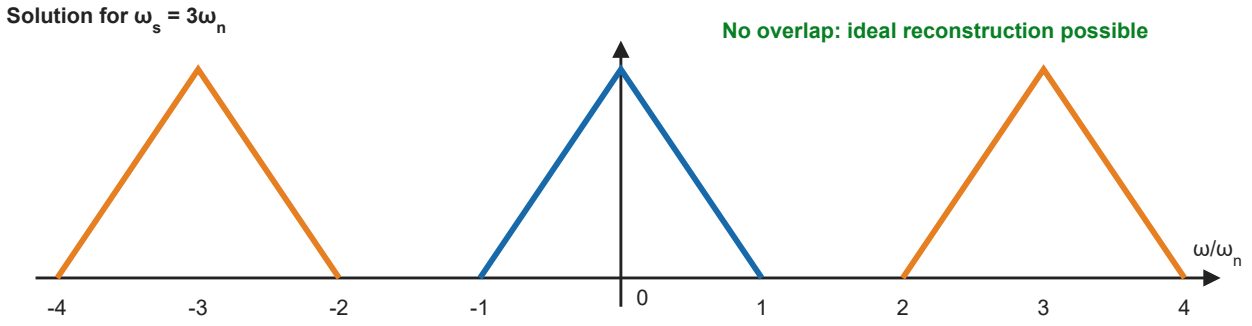
The central copy occupies

$$[-\omega_\mathrm{n}, \omega_\mathrm{n}].$$

The nearest shifted copies occupy

$$[2\omega_\mathrm{n}, 4\omega_\mathrm{n}] \quad (n = 1), \qquad [-4\omega_\mathrm{n}, -2\omega_\mathrm{n}] \quad (n = -1).$$

There is a gap of width ω_n between the baseband copy and each nearest replica. No copies overlap. Since $\omega_s = 3\omega_n > 2\omega_n$, the condition of the sampling theorem is satisfied and $v_a(t)$ can be recovered by selecting the central copy with an ideal reconstruction lowpass.



1.2 $\omega_s = 1.5\,\omega_n$

The nearest positive-frequency copy is supported on

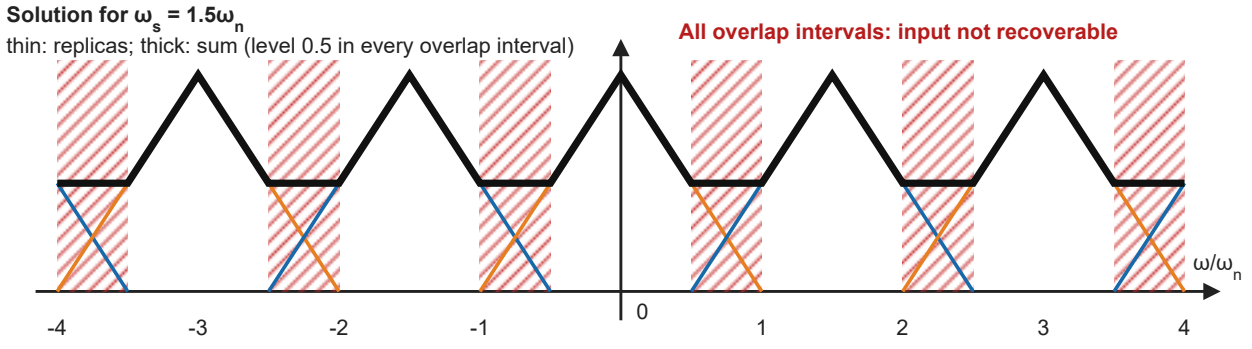
$$[0.5\,\omega_n, 2.5\,\omega_n],$$

and therefore overlaps the baseband copy on $[0.5\,\omega_n, \omega_n]$. The negative-frequency copy overlaps it on $[-\omega_n, -0.5\,\omega_n]$.

Since $\omega_s = 1.5\,\omega_n < 2\omega_n$, the sampling theorem is violated. Values from different parts of the original spectrum add in the same observed frequency regions. A reconstruction filter cannot determine which contribution came from which original frequency. The signal $v_a(t)$ therefore cannot be recovered, even with an ideal reconstruction filter.

The thin colored curves in the solution sketch show the individual replicas. The thick black curve shows their sum $V_i(j\omega)$, with the common factor $1/T$ omitted. For this particular triangular spectrum and the particular spacing $\omega_s = 1.5\,\omega_n$, the sum is constant at 0.5 within every overlap interval. This is a property of this example, not a general property of spectral aliasing.

Core explanation: aliasing is not merely the presence of spectral replicas. Aliasing is their overlap after sampling.



2. Demo prediction and observation (12-22 min)

1. Reducing ω_s moves the replicas closer together. They first touch at $\omega_s = 2\omega_n$ and overlap for $\omega_s < 2\omega_n$.
2. With $f_s = 8\,\mathrm{kHz}$, half the sampling frequency is $4\,\mathrm{kHz}$. The component at $6\,\mathrm{kHz}$ folds to

$$f_{\text{alias}} = |6 - 8|\,\mathrm{kHz} = 2\,\mathrm{kHz}.$$

In angular frequency: the replica $n = -1$ shifts $\omega = 2\pi \cdot 6\,\mathrm{kHz}$ to $\omega - \omega_s = -2\pi \cdot 2\,\mathrm{kHz}$; its mirror component at $+2\pi \cdot 2\,\mathrm{kHz}$ comes from the replica $n = +1$ of the negative-frequency component. The explanation by shifted replicas is preferred over memorizing the formula. 3. Without an anti-aliasing filter, the 6 kHz component enters the sampler and aliases to 2 kHz. An ideal anti-aliasing lowpass can remove it sharply.

A realizable filter requires a transition band and provides only finite stopband attenuation, so a reduced alias component remains.

Practical consequence: the useful signal band must normally end below $\omega_s/2$. The resulting guard band gives a realizable filter space in which to transition from passband to stopband.

Answers to the notebook discussion prompts

1. Why can a digital lowpass not distinguish an original 2 kHz component from a 6 kHz component sampled at 8 kHz?

At $f_s = 8\,\mathrm{kHz}$, the 6 kHz component is mapped to the same discrete-time frequency as a component at $-2\,\mathrm{kHz}$; for a real-valued signal its conjugate counterpart therefore appears at $+2\,\mathrm{kHz}$. Equivalently,

$$f_{\mathrm{alias}} = |6 - 8|\,\mathrm{kHz} = 2\,\mathrm{kHz}.$$

After sampling, the digital lowpass sees only the resulting discrete-time component at 2 kHz. The information indicating whether it originated from an analog component at 2 kHz or 6 kHz is no longer present. A digital filter can attenuate that component, but it cannot separate the two possible origins.

2. Why does a realizable anti-aliasing filter require a guard band?

A realizable analog filter cannot change from unity gain to zero gain at a single frequency. It needs a finite transition band between its passband and stopband. Therefore, the useful signal band must end below $\omega_s/2$, leaving frequency space in which the filter can reach the required stopband attenuation before spectral components would alias into the baseband.

3. Which observed distortion is irreversible, and which sample-and-hold effect is compensable?

Aliasing is irreversible because contributions from different analog frequencies have already been added at the same discrete-time frequency. Their individual origins cannot be recovered from the sampled sequence.

The deterministic sample-and-hold magnitude droop

$$\left|\mathrm{si}\left(\frac{\omega T}{2}\right)\right|$$

is, in principle, compensable wherever the si response is nonzero. Compensation may be implemented by digital pre-equalization or by a suitably shaped reconstruction filter. The linear phase factor $e^{-j\omega T/2}$ represents the known delay $T/2$.

4. How do the four frequency-dependent blocks combine, and when is the equivalent system valid?

With the sampling and converter convention used in this exercise,

$$H_{\mathrm{eff}}(j\omega) = H_{\mathrm{AA}}(j\omega)H\left(e^{j\omega T}\right)\frac{1}{T}H_{\mathrm{sh}}(j\omega)H_{\mathrm{r}}(j\omega), \qquad |\omega| < \frac{\omega_s}{2}.$$

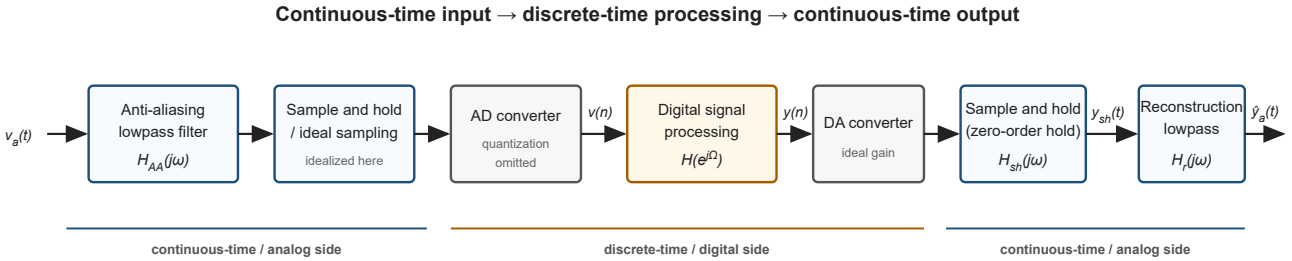
The factor $1/T$ from ideal impulse-train sampling cancels the factor T in H_{sh}. The description as a single equivalent analog LTI system requires

1. no aliasing on the input side, and
2. suppression of the spectral images on the output side.

With realizable filters these conditions hold only approximately, so H_{eff} is correspondingly an approximation. The detailed derivation follows in Section 3.4.

3. The complete AD-DSP-DA chain (22-35 min)

The block diagram used in the exercise follows the lecture notes (Part 2, slides 3 and 20-22):



Exercise model: impulse-train sampling contributes 1/T; converter gains are ideal; quantization is treated in Exercise 2.

The input sample-and-hold circuit and the AD/DA converter gains are idealized. Quantization is omitted. Consequently, these blocks do not add frequency-dependent factors to the equivalent response used in this exercise.

The amplitude convention is nevertheless fixed: the impulse-train sampling model from Section 1 contributes $1/T$ in the alias-free baseband, while the ideal DA converter has unit gain. The factor $1/T$ is constant and therefore not counted among the four frequency-dependent blocks.

3.1 Digital frequency mapping

At $\omega = \omega_\mathrm{n}$ and $\omega_\mathrm{s} = 3\omega_\mathrm{n}$,

$$\Omega_\mathrm{n} = \omega_\mathrm{n} T = 2\pi \frac{\omega_\mathrm{n}}{3\omega_\mathrm{n}} = \frac{2\pi}{3}.$$

The complete analog baseband $|\omega| \le \omega_\mathrm{n}$ maps to $|\Omega| \le 2\pi/3$.

3.2 Effect of the digital filter

Case 1: $\Omega_\mathrm{c} = 0.8\pi$

Since

$$0.8\pi > \frac{2\pi}{3},$$

the digital lowpass passes the complete triangular spectrum. Its equivalent analog cutoff is

$$\omega_\mathrm{c} = \frac{0.8\pi}{2\pi}(3\omega_\mathrm{n}) = 1.2\,\omega_\mathrm{n}.$$

The original signal band ends at ω_n, so it is unchanged.

Case 2: $\Omega_\mathrm{c} = 0.4\pi$

Here

$$0.4\pi < \frac{2\pi}{3},$$

and the equivalent analog cutoff is

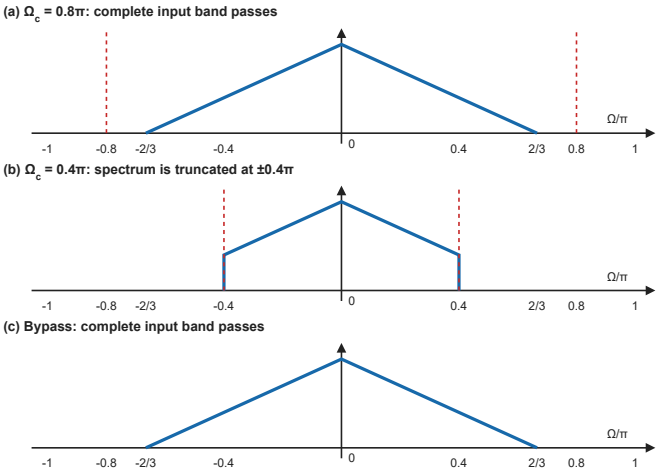
$$\omega_\mathrm{c} = \frac{0.4\pi}{2\pi}(3\omega_\mathrm{n}) = 0.6\,\omega_\mathrm{n}.$$

Only the portion $|\omega| \le 0.6\,\omega_\mathrm{n}$ passes. For an ideal brick-wall digital lowpass, the triangular spectrum is abruptly set to zero outside this interval; at the cutoff its value is $1 - 0.6 = 0.4$.

Case 3: $H(e^{j\Omega}) = 1$

The digital processing block does not modify the sampled sequence. Under alias-free sampling, the complete original baseband remains available. This does not make either the analog anti-aliasing filter or the analog reconstruction filter unnecessary: they operate on different sides of the sampler and converter and have different functions.

The three solution sketches are summarized below:



3.3 Sample-and-hold circuit and reconstruction

For

$$H_{\mathrm{sh}}(j\omega) = T\,\mathrm{si}\left(\frac{\omega T}{2}\right) e^{-j\omega T/2},$$

the factors have the following meanings:

- $T\left|\mathrm{si}(\omega T/2)\right|$: the magnitude response of the sample-and-hold circuit. Its magnitude decreases toward higher frequencies in the reconstructed baseband. This is the sample-and-hold (si) droop. At the band edge $\omega = \omega_{\mathrm{s}}/2$, $\mathrm{si}(\pi/2) = 2/\pi \approx 0.64$, i.e. about $-3.9\,\mathrm{dB}$. Within $|\omega| < \omega_{\mathrm{s}}/2$, the si term is positive; the absolute value is required for the general magnitude response.
- $e^{-j\omega T/2}$: a linear phase term; it represents a delay of $T/2$.
- $H_{\mathrm{r}}(j\omega)$: the analog reconstruction filter suppresses the spectral images above $\omega_{\mathrm{s}}/2$ produced by conversion back to continuous time and smooths the staircase-like signal $y_{\mathrm{sh}}(t)$.

The droop is deterministic and, where the si response is nonzero (within the baseband everywhere), can in principle be equalized. It is therefore fundamentally different from aliasing. The lecture (slide 22) suggests "pre-biasing" the reconstruction filter, i.e. letting H_{r} rise like $1/\,\mathrm{si}(\omega T/2)$ in the passband; alternatively, the inverse si characteristic can be included digitally in $H(e^{j\Omega})$. No compensation design is required in this exercise.

3.4 Equivalent analog system

With the stated impulse-train sampling convention, ideal sampling first contributes $1/T$ in the alias-free baseband. The ideal DA converter has unit gain. Therefore, for alias-free sampling and baseband reconstruction,

$$H_{\mathrm{eff}}(j\omega) = H_{\mathrm{AA}}(j\omega)\, H\left(e^{j\omega T}\right) \frac{1}{T} H_{\mathrm{sh}}(j\omega)\, H_{\mathrm{r}}(j\omega), \qquad |\omega| < \frac{\omega_{\mathrm{s}}}{2}.$$

Substituting the sample-and-hold response gives

$$H_{\mathrm{eff}}(j\omega) = H_{\mathrm{AA}}(j\omega)\, H\left(e^{j\omega T}\right) \mathrm{si}\left(\frac{\omega T}{2}\right) e^{-j\omega T/2}\, H_{\mathrm{r}}(j\omega), \qquad |\omega| < \frac{\omega_{\mathrm{s}}}{2},$$

and $H_{\mathrm{eff}}(j\omega) = 0$ for $|\omega| \geq \omega_{\mathrm{s}}/2$.

Interpretation of the factors:

- H_{AA}: limits the analog input bandwidth before sampling and prevents aliasing.
- $H(e^{j\omega T})$: performs the intended digital signal processing; its digital frequency argument is obtained from $\Omega = \omega T$.
- H_{sh}/T: introduces the si magnitude droop and a delay of $T/2$.

- H_r: removes the output spectral images.

The factor $1/T$ comes from the ideal impulse-train sampling model used in this exercise. It cancels the factor T in H_sh. This matches slide 21 of the lecture, where an ideal reconstruction filter with unit gain gives $|\hat{Y}_\mathrm{a}(j\omega)| = |Y_\mathrm{a}(j\omega)|\,|\,\mathrm{si}(\omega T/2)|$. Without the hold circuit, the lecture's ideal reconstruction lowpass instead carries the gain T (slide 17, $H_\mathrm{r}(j\omega) = T\,\mathrm{rect}(\omega/\omega_\mathrm{s})$). Other consistently normalized converter models may distribute constant gains differently, but within the convention specified here the cancellation must be shown explicitly.

Conditions of validity. The chain is equivalent to a single LTI system with frequency response $H_\mathrm{eff}(j\omega)$ only if both of the following hold:

1. **Input side, no aliasing:** $V_\mathrm{a}(j\omega)H_\mathrm{AA}(j\omega) = 0$ for $|\omega| \geq \omega_\mathrm{s}/2$. Otherwise different input frequencies contribute to the same baseband frequency, and the output cannot be described by a one-to-one transfer function of the original input.
2. **Output side, no images:** $H_\mathrm{r}(j\omega) = 0$ for $|\omega| \geq \omega_\mathrm{s}/2$. The sampled-and-held signal $y_\mathrm{sh}(t)$ contains images around $k\omega_\mathrm{s}$, weighted by the si function. If they are not removed, $\hat{y}_\mathrm{a}(t)$ contains frequencies $\omega + k\omega_\mathrm{s}$ that are not present in the input; the chain is then linear but only periodically time-varying (period T) and has no single frequency response.

With realizable filters, both conditions hold only approximately; H_eff is then an approximation whose accuracy is set by the stopband attenuation of H_AA and H_r.

4. Expected oral-examination concepts (35-43 min)

A strong explanation should distinguish the following:

- Before sampling: anti-aliasing is preventive.
- After sampling: digital filtering can shape the correctly represented discrete-time spectrum but cannot identify and separate already aliased contributions.
- At D/A conversion: conversion of the discrete sequence to a continuous-time impulse train produces periodic spectral images. The sample-and-hold circuit then weights these images with its si-shaped frequency response and introduces the si droop.
- After D/A conversion: the reconstruction filter suppresses images.
- Irreversible errors: overlap of spectral replicas during sampling. Outside the model of this exercise, quantization in the AD converter is also irreversible (Exercise 2).
- In-principle compensable effects: deterministic passband shaping such as the si droop or nonzero filter attenuation.
- Validity of H_eff: no aliasing at the input and no remaining images at the output.

5. Take-away (43-45 min)

1. Aliasing must be prevented before or during sampling by sufficient sampling rate and analog band limitation. Once spectral components overlap, their origins cannot be uniquely recovered.
2. The equivalent analog response is the product of the anti-aliasing, mapped digital, sample-and-hold, and reconstruction responses, provided that sampling is alias-free and the reconstruction filter removes all images.