# ADSP - Exercise 1: From Analog Input to Analog Output

## Learning objectives

After this exercise, you should be able to

- explain aliasing using shifted spectral replicas,
- distinguish the roles of the anti-aliasing, digital, and reconstruction filters,
- map an analog angular frequency $\omega$ to the normalized digital frequency $\Omega = \omega T$,
- interpret the magnitude and phase response of the sample-and-hold circuit at the output, and
- formulate and explain the equivalent analog frequency response of an AD-DSP-DA chain, including its conditions of validity.

**Guiding question:** What is irreversibly lost during sampling, what can still be changed or compensated digitally, and how can the complete AD-DSP-DA chain be described?

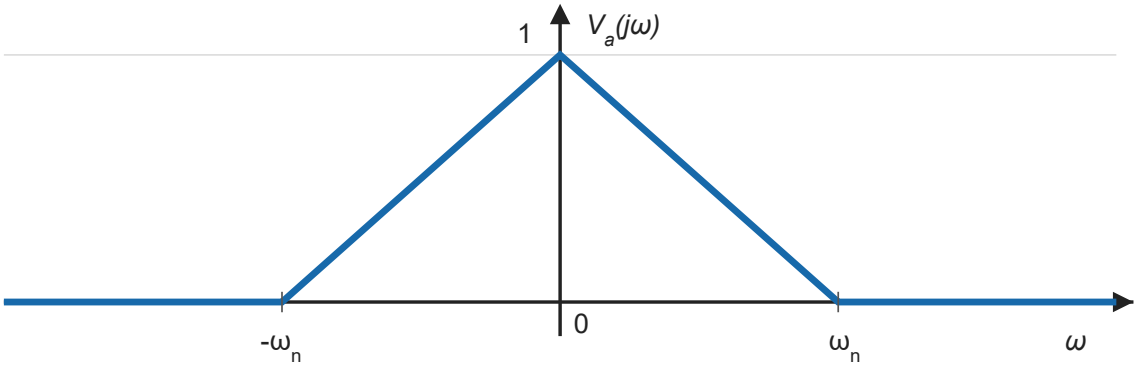
**Notation (lecture notes, Part 2, slides 3-22):** $\omega$ analog angular frequency, $T$ sampling period, $\omega_\mathrm{s} = 2\pi/T$ sampling frequency, $\Omega = \omega T$ normalized digital frequency, $v_\mathrm{a}(t) \circ\!\!-\!\!\bullet\, V_\mathrm{a}(j\omega)$ continuous-time signal, $v(n) \circ\!\!-\!\!\bullet\, V(e^{j\Omega})$ sequence, $\mathrm{si}(x) = \sin(x)/x$. Frequencies in hertz are written as $f = \omega/(2\pi)$ and $f_\mathrm{s} = 1/T$.

## 1. Reactivation: sampling and aliasing

Consider a real-valued continuous-time signal $v_\mathrm{a}(t)$ with highest frequency $\omega_\mathrm{n}$ and the symmetric spectrum

$$V_\mathrm{a}(j\omega) = \mathrm{tri}\left(\frac{\omega}{\omega_\mathrm{n}}\right) = \begin{cases} 1 - \dfrac{|\omega|}{\omega_\mathrm{n}}, & |\omega| \le \omega_\mathrm{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Reference spectrum:




Ideal sampling with the impulse train $p(t) = \sum_n \delta_0(t - nT)$ produces the ideally sampled signal $v_\mathrm{i}(t) = v_\mathrm{a}(t)\, p(t)$ with the spectrum

$$V_\mathrm{i}(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} V_\mathrm{a}\big(j(\omega - n\omega_\mathrm{s})\big).$$

As in the lecture notes, $n$ denotes the replica index in this sum. For the sketches below, the constant amplitude factor $1/T$ may be ignored. Concentrate on the position, shape, and overlap of the spectral replicas.

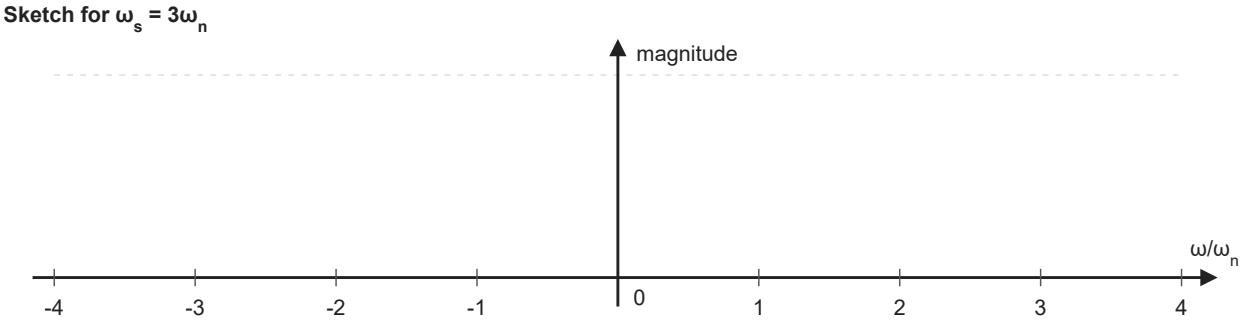
### 1.1 Alias-free sampling

Sketch $V_i(j\omega)$ for

$$\omega_s = 3\omega_n.$$

Mark the original baseband spectrum and at least the replicas for $n = \pm 1$. State whether $v_a(t)$ can be recovered using an ideal reconstruction filter.

Use the coordinate system below for your sketch:



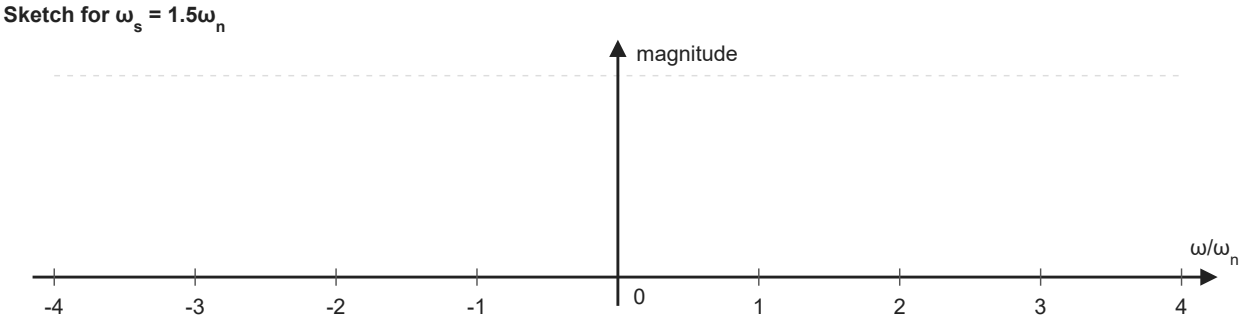

### 1.2 Sampling with aliasing

Repeat the sketch for

$$\omega_s = 1.5\,\omega_n.$$

Mark all overlap regions. State whether $v_a(t)$ can be recovered using an ideal reconstruction filter.

Use the coordinate system below for your sketch:




## 2. Demo prediction and observation

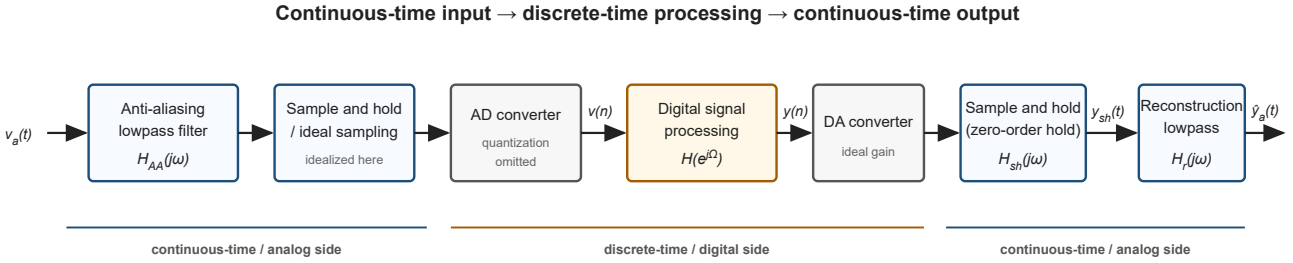
Before each demonstration, predict the result. Then compare your prediction with the displayed spectrum.

1. What changes when $\omega_s$ is reduced continuously from $3\omega_n$ to $1.5\,\omega_n$?
2. Consider an audio signal containing tones at $f = 1\,\mathrm{kHz}$ and $f = 6\,\mathrm{kHz}$. At which frequency will the $6\,\mathrm{kHz}$ component appear when the signal is sampled at $f_s = 8\,\mathrm{kHz}$? Explain using a spectral sketch or folding about half the sampling frequency.
3. Compare the following anti-aliasing options in the demo:
   - no anti-aliasing filter,
   - an ideal lowpass,
   - a realizable-lowpass magnitude model with a transition band (phase is not considered here).

Record one practical consequence of the transition band of a realizable filter.

## 3. The complete AD-DSP-DA chain

Consider the following processing chain. Block and signal names follow the lecture notes (Part 2, slides 3 and 20-22); the symbol $H_{AA}$ for the anti-aliasing filter is introduced here.




Exercise model: impulse-train sampling contributes 1/T; converter gains are ideal; quantization is treated in Exercise 2.

The input sample-and-hold circuit and both converter gains are idealized. Quantization is not considered here. Therefore, only the four frequency-dependent blocks $H_{\mathrm{AA}}(j\omega)$, $H(e^{j\Omega})$, $H_{\mathrm{sh}}(j\omega)$, and $H_{\mathrm{r}}(j\omega)$ appear in the equivalent analog frequency response derived below.

For amplitude scaling, use the impulse-train sampling convention introduced in Section 1: in the alias-free baseband, ideal sampling contributes the constant factor $1/T$. The ideal DA converter has unit gain. This $1/T$ factor is not an additional frequency-dependent block.

### 3.1 Digital frequency mapping

For alias-free sampling with $\omega_{\mathrm{s}} = 3\omega_{\mathrm{n}}$, determine the normalized digital frequency $\Omega_{\mathrm{n}}$ corresponding to the analog band edge $\omega_{\mathrm{n}}$:

$$\Omega = \omega T = 2\pi \frac{\omega}{\omega_{\mathrm{s}}}.$$

### 3.2 Effect of the digital filter

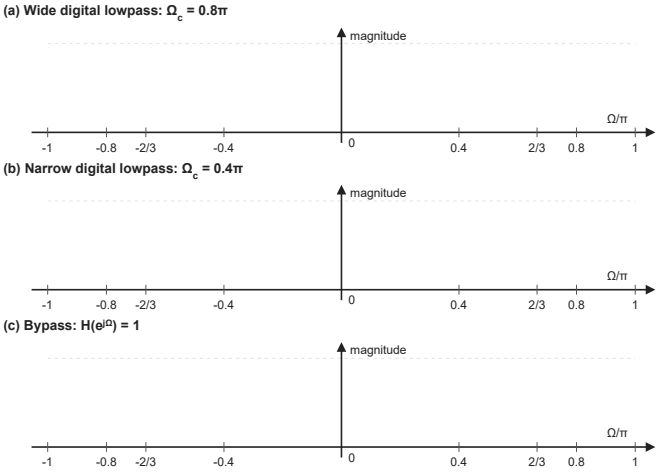
Assume an ideal digital lowpass $H(e^{j\Omega})$ with cutoff $\Omega_{\mathrm{c}}$. Sketch the baseband spectrum after digital filtering for each case:

1. $\Omega_{\mathrm{c}} = 0.8\pi$,
2. $\Omega_{\mathrm{c}} = 0.4\pi$,
3. $H(e^{j\Omega}) = 1$.

For each case, state whether the original triangular baseband spectrum is unchanged or partly removed. Relate $\Omega_{\mathrm{c}}$ to the corresponding analog cutoff frequency

$$\omega_{\mathrm{c}} = \frac{\Omega_{\mathrm{c}}}{T} = \frac{\Omega_{\mathrm{c}}}{2\pi}\,\omega_{\mathrm{s}}.$$

Use the three coordinate systems below:



### 3.3 Sample-and-hold circuit and reconstruction

As in the lecture notes, the output sample-and-hold circuit (zero-order hold) has the impulse response $h_{\mathrm{sh}}(t) = \mathrm{rect}\big((t - T/2)/T\big)$ and the frequency response

$$H_{\mathrm{sh}}(j\omega) = T\ \mathrm{si}\left(\frac{\omega T}{2}\right) e^{-j\omega T/2}.$$

Note: slide 22 of the lecture labels the same curve $T\,\mathrm{sinc}(\omega T/2)$; there, "sinc" denotes the function $\mathrm{si}(x) = \sin(x)/x$.

Interpret separately:

1. the magnitude response

$$|H_{\mathrm{sh}}(j\omega)| = T\left|\mathrm{si}\left(\frac{\omega T}{2}\right)\right|,$$

2. the factor $e^{-j\omega T/2}$,
3. the task of the reconstruction filter $H_{\mathrm{r}}(j\omega)$.

Classify the sample-and-hold (si) droop as irreversible or, in principle, compensable.

### 3.4 Equivalent analog system

Combine the four frequency-dependent blocks to obtain the equivalent analog frequency response $H_{\mathrm{eff}}(j\omega)$ from $v_{\mathrm{a}}(t)$ to $\hat{y}_{\mathrm{a}}(t)$ in the baseband $|\omega| < \omega_{\mathrm{s}}/2$. Use the mapping $\Omega = \omega T$ and carry along the sampling factor $1/T$ specified above. Show explicitly how it combines with the factor $T$ in $H_{\mathrm{sh}}(j\omega)$.

For every factor in your equation, state its physical purpose or effect. Then state the conditions on the input side **and** on the output side under which the complete chain behaves like a single LTI system with this frequency response.

## 4. Peer oral examination

Work in pairs. Use examination cards A and B.

- 3 min: Student A examines student B using card A.
- 3 min: Change roles; student B examines student A using card B.
- 2 min: Give feedback using the checklist on the cards.

## 5. Take-away

Formulate two statements:

1. What must be done to prevent irreversible aliasing?
2. Which blocks determine the equivalent analog response of an AD-DSP-DA chain, and under which conditions does this description hold?